Report of the Commission on Mathematics

Program for college preparatory mathematics

College Entrance Examination Board

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College Entrance Examination Board
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presents

a nine-point program

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Report of the Commission on Mathematics

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Foreword

The pride with which the College Entrance Examination Board presents the report of the Commission on Mathematics is the product of the belief, shared by the Trustees and staff of the Board, that the Commission's work is an extraordinary and inspiring example of cooperative educational effort.

The very creation of the Commission was the result of a combination of concerns and interests expressed by a number of Board groups: the Examiners in Mathematics who requested a study of the secondary school mathematics curriculum; the Committee on Examinations, which recognized the need for exploratory action; and the Trustees, who authorized the appointment of the Commission. Prime movers in giving life to the Commission by strongly recommending its creation were the chairman of the Committee on Examinations, Dean Albert E. Meder, Jr., of Rutgers University, and the two distinguished educators with whom he consulted, Professor Howard F. Fehr of Teachers College, Columbia University, and Professor Albert W. Tucker of Princeton University.

The cooperation and contribution of many individuals and institutions continued after the Commission was formally established and began its work in August 1955 under the chairmanship of Professor Tucker. The Educational Testing Service provided space for a headquarters unit and released from its own staff Robert Kalin, who rendered valuable service as Executive Assistant of the Commission during the first year of its life. Somewhat later, the Carnegie Corporation joined in assistance of the Commission's program by providing funds toward the support of the Commission's activities. At about the same time, Rutgers University granted Dean Meder a leave of absence during 1957, later extending the period by six months, to permit him to serve as Executive Director of the Commission, an exceptional demonstration of institutional generosity. When Dean Meder's return to his post could no longer be delayed, continuation of the Commission's work was again assured in the most highly cooperative fashion by Kent School's selfless reaction to the Board's request that its mathematics department head, Robert E. K. Rourke, serve as Executive Director for one year.

During the life of the Commission, Princeton University and Kent School graciously provided congenial facilities for the meetings of the Commission and for the effective work of its writing groups.

Above all, of course, the individual members of the Commission must be credited with an unusual group accomplishment. Their recommendations represent not only the completion of a task of heroic proportions but a notably successful attempt to work in concert.

In addition, the following persons gen-

erously participated in the Commission's deliberations by attending one or more of its meetings: Frank B. Allen, chairman, Secondary School Curriculum Committee, National Council of Teachers of Mathematics; Max Beberman, director, University of Illinois Committee on School Mathematics Project; Edward G. Begle, director, School Mathematics Study Group; William Betz, supervisor of mathematics (retired), Rochester, New York; R. S. Burington, chief mathematician, Bureau of Ordnance, Department of the Navy; S. S. Cairns, professor of mathematics, University of Illinois; Robert L. Davis, member, Committee on the Undergraduate Program, Mathematical Association of America; George E. Hay, professor of mathematics, University of Michigan; Saunders MacLane, professor of mathematics, University of Chicago; John R. Mayor, director of education, American Association for the Advancement of Science; Eugene Nichols, associate professor of education, Florida State University; David A. Page, assistant professor of education, University of Illinois; G. Baley Price, president, Mathematical Association of America; Thomas D. Reynolds, Assistant Program Director for Summer Institutes, National Science Foundation; and Herbert E. Vaughan, associate professor of mathematics, University of Illinois.

Still others who gave valuable time and invaluable assistance by working with Commission members in the writing groups that produced the large quantity of material illustrating the Commission's recommendations are: Julius H. Hlavaty, chairman, department of mathematics, DeWitt Clinton High School, New York, New York; Robert Kalin, instructor in education, Florida State University; John J. Kinsella, professor of education, New York University; Richard S. Pieters, chairman, department of mathematics, Phillips Academy, Andover, Massachusetts; Donald E. Richmond, Wells professor of mathematics, Williams College; and Myron F. Rosskopf, professor of mathematics, Teachers College, Columbia University.

Although the Commission did not lack for ambition or daring in taking as its province the field of college preparatory mathematics, it did so modestly and quietly, without seeking any mechanisms for enforcing its recommendations beyond those of reasonable presentation and argument. It is, therefore, all the more gratifying to realize that the Commission's work has been effective and that it is being carried forward by other groups which have the interest, the funds, and the capacity to translate the proposals which appear in this report into courses of study, textbooks, teachers' guides, reference materials, and the other indispensable tools of instruction that are necessary to bring about any major curricular change. It is the hope of the Board that, in due course, the Commission's recommendations will be integrated with those of other groups to improve the study of mathematics from the first grade to the graduate school.

Frank H. Bowles, *President*
College Entrance Examination Board

Assignment: the Commission's role

Some years ago the Mathematics Examiners of the College Entrance Examination Board—the distinguished school and college teachers who are responsible for the Board's entrance examinations in mathematics — began to feel increasing concern about the curriculum they were testing. The Examiners wondered if it were appropriate, in the second half of the twentieth century, for an examination in secondary school advanced mathematics to be devoted, in approximately equal parts, to trigonometry, advanced algebra, and solid geometry. Did the tests accurately reflect what the schools were teaching? If so, was this what the schools should be teaching?

Knowing that some excellent schools and teachers already were introducing work of a different character, the Examiners wanted to know what kind of mathematics *should be* studied by today's American youth capable of going on to college work. The importance of this question is reflected in the fact that the group of students who take four years of high school mathematics includes almost all those who later study engineering, science, and mathematics in college.

Moreover, the Examiners were aware that curricular changes were already under way in some colleges. What were the implications of these changes for the secondary schools? Such questions went far beyond the matter of the pertinence of the mathematics testing of the College Board: they were concerned with the appropriateness and aptness of the mathematics curriculum itself.

The Examiners therefore referred these questions to the Board's Committee on Examinations, a group of school and college officers responsible for studying the Board's testing program and making recommendations concerning it. This committee soon came to share the views of the Mathematics Examiners. It felt that curricular reform in secondary school mathematics was long overdue, and that the Board, as an agency representing both colleges and secondary schools, could and should use its influence to improve the current situation. The committee recommended the establishment of a small but widely representative Commission composed of mathematicians and school and college teachers of mathematics who would be asked to review the existing secondary school mathematics curriculum, and to make recommendations for its modernization, modification, and improvement. The Board accepted the recommendation and the Commission on Mathematics was appointed in 1955.

This action of the Board constituted a frank attempt to influence the secondary school curriculum. Influence is being exer-

cised, however, only through the work of the mathematicians, teachers of mathematics, and teachers of teachers of mathematics who are on the Commission, and not at all through direct action of the Board or its standing committees, which are composed largely of administrative officers—deans, principals, and admissions officers. Further, the Board considered and rejected the possibility of a sudden change in its mathematics examinations in favor of an exploration of curriculum improvement which could become the guide for a gradual change of examination practice.

In other words, the Board's action was designed to bring to bear on the problem of the mathematics curriculum the considered judgment of persons professionally concerned with it. Thus, in setting up the Commission on Mathematics, the Board remained true to its highest function as an agency in which school and college representatives are brought together to discuss, develop, and implement policies with respect to students' transition from school to college.

The Commission has been particularly concerned with the importance of secondary school mathematics as a foundation for college work. Approximately 80 per cent of the member colleges of the College Board include some study of mathematics among their entrance requirements, and all of them welcome such study in secondary school.

Not all of us on the Commission hold precisely the same views on education in general, or on secondary school mathematical education in particular. We represent different backgrounds, diverse professional experiences, and a variety of institutional affiliations. Some of us have dealt largely with highly selected groups of students, others with more heterogeneous groups. However, we all believe in certain principles of education, many of which find expression in this report and have been applied in working out our recommendations.

The establishment of the Commission has enabled a group of professional men and women to subject an important and urgent problem to intensive scrutiny and study, and to make recommendations for its solution. These recommendations, in the form of a proposed new program for secondary school mathematics, we now place in the hands of those actively engaged in secondary education throughout the nation. Only they can translate them into action. Acceptance of the recommendations of the Commission can come about only if these proposals develop the power of an idea whose time is ripe.

1. Orientation: an urgent need for curricular revision

Mathematics is a living, growing subject. The vitality and vigor of present-day mathematical research quickly dispels any notion that mathematics is a subject long since embalmed in textbooks. Mathematics today is in many respects an entirely different discipline from what it was at the turn of the century. New developments have been extensive; new concepts have been revolutionary. The sheer bulk of current mathematical development is staggering.

Creation of new mathematics

The American Mathematical Society, the learned society in this country devoted primarily to the advancement of mathematical research, publishes a journal entitled *Mathematical Reviews,* in which brief notice is given of new mathematical results. No proofs are included, nor any expositions, but merely critical factual reports—and still the journal runs to some 1,200 large, double-column pages each year.

Also, the Society holds meetings at which research results are communicated. Over 800 papers are presented annually at such meetings. The Society's membership in the United States and Canada alone numbers more than 5,000. When one realizes that mathematicians are active all over the world, he begins to see why the annual production of new mathematics is formidable.

One inevitable result of this explosive development of mathematics has been the creation of new subject matter. Such fields as mathematical logic, probability and statistical inference, topology, and modern abstract algebra are largely or wholly the products of recent mathematical research. Mathematical logic, for example, was little known to most mathematicians a generation ago. But by 1936 its development was sufficiently extensive, and interest in it sufficiently widespread, to warrant the formation of an international learned society, the Association for Symbolic Logic, devoted exclusively to this branch of mathematics. The Association publishes some 200 pages of research annually.

Similarly, the field of probability and statistical inference has been extensively developed in recent years. In this field, too, a research society and research journal have been established in this country: the Institute of Mathematical Statistics, and the *Annals of Mathematical Statistics,* now in its twenty-ninth year. At least 10 other journals,

in English, are entirely devoted to the theory and applications of probability and statistics.

These exciting new mathematical developments also receive popular notice. For example, articles on "The New Mathematics" and "The New Uses of the Abstract" by G. A. W. Boehm recently have appeared in *Fortune,* in the issues of June and July 1958.

Reorganization of older mathematics

Another result of the recent growth of mathematics has been the reorganization, extension, and transformation of parts of the older mathematics.

Algebra is a branch of mathematics that has been transformed by the mathematical research of the last quarter-century. A first-year graduate school course in algebra today bears little or no resemblance to that of 30 years ago. The key concepts of the earlier course would now be subsumed under newer and broader concepts that had not been formulated in 1928. Yet the corresponding course of 30 years ago differed little or not at all from its predecessor of 1900.

The essence of this transformation is essentially that algebra is now thought of as the study of mathematical structure, or "pattern." For example, high school algebra deals extensively with systems of numbers: positive and negative integers, rational numbers, real numbers, and complex numbers. One rule of combination of numbers is multiplication. In all of these systems, two of the patterns of multiplication are given by the commutative and associative laws:

$$a \times b = b \times a$$

and

$$(a \times b) \times c = a \times (b \times c).$$

Today's algebra encompasses as well systems in which the elements are not numbers at all. And in such systems familiar patterns may not appear—for example, the commutative and associative laws do not hold in some algebras.

Earlier mathematicians often looked upon algebra from a manipulative point of view. Skill in performing the operations within the system frequently was the goal of instruction, rather than an understanding of the properties of the system. The contemporary point of view, while not discounting the manipulative skills necessary for efficient mathematical thought, puts chief emphasis on the structure or pattern of the system and on deductive thinking.

The nature of contemporary mathematics

The difference in point of view between the older approach and the contemporary conception is well put by W. W. Sawyer: "The mathematician of older times asked, 'Can I find a trick to solve this problem?' If he could not find a trick today, he looked for one tomorrow. But . . . we no longer assume that a trick need exist at all. We ask rather, 'Is there any reason to suppose that this problem can be solved with the means we have at hand? Can it be broken up into simpler problems? What is it that makes a problem soluble, and how can we test for solubility?' We try to discover the nature of the problem we are dealing with."[1]

Today, developments in mathematics are concerned with such patterns of thought and

[1] *Prelude to Mathematics* (Baltimore: Pelican Books, Penguin Books, Inc., 1955), p. 214.

such insights into meaning. Thus, contemporary mathematics is characterized by: (1) a tremendous development quantitatively; (2) the introduction of new content; (3) the reorganization and extension of older content; and (4) renewed, increased, and conscious emphasis upon the view that mathematics is concerned with abstract patterns of thought.

New applications of mathematics

Not only in pure mathematics are great vitality, new content, and spectacular growth to be found. The same developments are seen in recent applications of mathematics. Mathematics is extending itself, often with dramatic results, into fields in which until recently it had not been used.

There are two aspects to this greatly expanded application of mathematics: first, the new uses of the older mathematics; second, the uses of the newer mathematics.

New uses of older mathematics

Some idea of the scope and nature of the recent application of mathematics long since known is conveyed by the following statement prepared for the Commission by M. S. Longuet-Higgins, a British expert on fluid dynamics:

"The basic mathematical ideas and methods that have been known for centuries still are as valid, as exciting, and as practically useful as ever. Especially is this true in the fields of calculus and mathematical analysis. Many of the most striking advances of our age, for example the development of supersonic flight and the launching of earth satellites, depend directly on an expert application of fluid dynamics, to which calculus is absolutely fundamental. Current research in geophysics—in the upper atmosphere (ionosphere), in the oceans (discovery of deep ocean currents), and in the interior of the earth (theories of the magnetic field)—are closely associated with advances in mathematical theory. New achievements in mechanical engineering, in automatic control (automation), and in radar communication would be unthinkable but for the work of analysts (such as Norbert Wiener) and applied mathematicians (such as Claude Shannon). To all such workers calculus is a basic tool, and analysis is part of their very thought.

"The close connection between mathematics and physics also should not be forgotten. Most of the great physical discoveries have come from skilled use of mathematical technique. Newton's law of gravitation (on which the Sputniks and Explorers depend) was established by combined use of calculus and physical observation. The more refined Einstein law of gravitation is stated in terms of differential geometry. Theories of the atom are described in terms of Schroedinger's differential equation. A satisfactory theory of the atomic nucleus, still to be formulated, almost certainly will come in a mathematical form.

"Therefore, while the growing character of mathematics may be emphasized, the permanence of traditional mathematical ideas must not be forgotten. Mathematics may be compared to a great tree, ever putting forth green shoots and new branches, still nevertheless, having the same firm trunk of established knowledge. The new shoots are evidence of life, the trunk is essential for the support of the whole."

Uses of new mathematics

In addition to the impressive array of new uses to which the older mathematics is now put, there are also astonishing and rapidly expanding fields of application for contemporary mathematics. Mathematics is no longer reserved for the use of engineers and physical scientists, even though a great many of its applications are still in their hands.

Mathematics and mathematical techniques and modes of thought permeate our lives and activities today to an extent that relatively few people yet perceive. Previous ages regarded mathematics as a worthy handmaiden of science and engineering. Today mathematics exerts actual leadership in natural science, social science, business, industry, and other fields of application.

Mathematics in the social sciences: One of the most important of these new fields of application is that of the social sciences. Social scientists have frequent occasion to deal with large masses of data, usually obtained by sampling. With respect to such data, problems of presentation, description, and inference must be considered. How can extensive data be organized to show essential features and to reveal relationships without distortion? How can such a mass of data be accurately summarized without neglecting its dispersion? Under what conditions can inferences be drawn from the data, and how reliable are the inferences? Such mathematical questions are dealt with in the contemporary theory of probability and statistical inference.

Areas of the field of psychology increasingly are using quantitative methods, in the form of statistics and other advanced mathematical techniques. The same is true in sociology—in population and social studies concerning the individual and the community.

While economists for some time have found frequent uses for mathematics, especially calculus, in their work, recent years have seen these undergo a startling increase. This increase is reflected not only in the number and variety of problems attacked mathematically, but in the bewildering diversity of mathematics employed. And this is by no means a one-way street. Economics is contributing problems that lead to the development of new mathematics. The theory of games is an example of a branch of mathematics suggested by problems in economics related to competition and cooperation.

Economists have directed much research toward quantitative interrelations among various sectors of industry, toward constructing models of dynamic processes in the national economy, and toward predictions of the behavior of large economic complexes.

Economics and direct industrial applications overlap in the use of mathematical methods to solve complicated problems of management. In this field, sometimes called "operations research," are found specific instances of the application of the theory of games, of input-output analysis, and of programming for problems in machine loading, distribution, allocation, and scheduling.

The Social Science Research Council has set up a committee on the mathematical training of social scientists, and summer seminars have been conducted to help provide training in mathematics for persons active in social science research. Many research projects and organizations in social science have engaged mathematicians either as regular staff members or as consultants.

Mathematics in industry: The use of statistics has expanded greatly in industry. This is especially true in the design and analysis of industrial research experiments, and in statistical quality control and sampling theory. The techniques of statistical quality control, now extensively used in mass production industries, are essentially mathematical answers to questions of reliability of inference.

Samples of products being made are withdrawn for inspection or testing. What inferences as to the quality of the total production can be drawn from the results of the tests of such samples? Are defective items sufficiently few in number to justify acceptance of the product by the purchaser? What is the life expectancy of the product? Are defects, when they occur, caused by chance, by inferior raw material, or by faulty workmanship? Are the defective items the product of particular machines or particular workers? How reliable are the answers to these questions? The techniques of statistical quality control – techniques that are an application of contemporary mathematics—have been developed precisely to solve such problems.

Other applications of contemporary mathematics are found in the design of high-speed computing machines, in the communications industry, and in the theory of learning. The importance of these developments is well emphasized by the following statements quoted from a pamphlet entitled *Mathematics in an Industrial Economy.*[2]

"The growing volume of knowledge and the complexity of our social organization have made necessary an incredible expansion of the application of logical and quantitative methods. Here the automatic computer has greatly facilitated and extended the usefulness of mathematics. The electronic computers and data-processing systems make logical and analytical investigations possible on a scale undreamed of only a few decades ago. . . .

". . . Mathematics offers a medium for expressing complicated and logical relationships. It provides a framework for organizing masses of data and information. Its concepts and symbols display a dynamic dependence of the performance of the whole upon the behavior of its parts."

Significance for the secondary school curriculum

The fact that the new mathematics of the past 50 or 100 years is playing an ever-increasing role in our society is of special importance for the work of the Commission. It has profound implications with respect to curricular revision.

The Commission is concerned, as was the committee of examiners whose recommendations led to its appointment, over the fact that the present secondary school mathematics curriculum has lagged behind the growth and uses of the subject. The development of mathematics and the broadening of its applications have outrun the curriculum. Another way of putting this is to say that the present curriculum rests on a static rather than on a dynamic concept of mathematics. An analogy may help to dramatize the difference.

As a city grows, it becomes increasingly difficult to find adequate transportation from the center of the city to the outlying areas and the suburbs. The center is still the core

[2] (Detroit, Mich.: Industrial Mathematics Society, 1955 [revised, 1958]), p. 6.

of the city, but the streets are too narrow and too congested for the newer sections to be reached as quickly as the needs of the residents require. For a time, systems of traffic lights and one-way streets suffice; but ultimately these patchwork methods, too, are found to be inadequate. Then there is constructed a limited-access freeway or expressway from the heart of the city to outlying points, bringing the newer regions effectively closer to the core.

Precisely this process is what the growth of mathematics demands—namely, that new and more efficient routes be found in the foundations of the subject as laid in secondary school to the newer territory of modern mathematics, in order that students may penetrate these newer territories without laboriously traversing all of the older content. In the process some obsolete or obsolescent material will be dropped, as will some material whose omission will cause regret. While still of value, it is of lesser value than the objective of attaining an understanding of the spirit, method, and content of contemporary mathematics. Traditional mathematics—algebra, geometry, trigonometry—is still the great core of the subject, but the distance between the core and the newer developments must be shortened by new methods of approach.

Fortunately, much of the new mathematics suitable for high school is no more difficult, though of course less familiar, than a large part of the older mathematics. This is not surprising. A new topic must of necessity start with ideas that are in some sense elementary, whereas the further development of older topics cannot help but presuppose familiarity with a considerable amount of technical material. Intricate ideas are more likely to be found in the extensive development of any subject than in its beginnings.

It is fortunate that this is so because, as has been suggested, the scope of recent developments in mathematics and in its applications requires that schools and colleges recognize the existence of the new material and its importance. They should include in their courses of instruction at least selections from it or introductions to it. It is entirely feasible to accomplish this since much of the material to be introduced has the added advantage of being in some ways simpler than the traditional content.

Substantial changes in the mathematical curriculum are thus long overdue. But a mere change of subject matter is not sufficient. A poor curriculum well taught is better than a good curriculum badly taught. A good curriculum well taught is the only acceptable goal. With adequate guidance, students will seek courses in mathematics (and in anything else) that present significant, challenging subject matter, interestingly and meaningfully taught. The quality of the curriculum and the skill of the teacher both are vital: this combination alone can effect the kind of improvement in mathematical instruction that the United States must have.

Demand for improved instruction

There is abundant evidence of public dissatisfaction with mathematics programs, curricula, and instruction. Many writers have claimed that the general quality of mathematics teaching is in need of improvement, and that the level of mathematical competence of American high school graduates is low. Colleges complain that entering fresh-

men are poorly prepared in mathematics. The requirements in mathematics for high school graduation in many states are exceedingly limited. It is alleged that too small a percentage of high school students is enrolled in mathematics courses, and that many students dislike mathematics. Complaints such as these became very widespread after the launching of the first earth satellite.

It is not appropriate in this report to examine these popular beliefs in detail. No useful end would be served here by an attempt to determine statistically the proportion of secondary school students who dislike mathematics, or how they acquired their dislike. Nor would it assist the Commission, in dealing with its assigned task, if it were to become involved in a consideration of the percentage of students studying a particular course in mathematics today, as compared with some past date. Such a study would not be directly germane to the work of the Commission, nor would a discussion of minimum graduation requirements for mathematics in the high schools of the several states.

If the nation is indeed faced with low levels of mathematical competence—and in fact it is—the Commission thinks that part of the answer may be found in increasing the amount of time devoted to the study of mathematics (in both homework and schoolwork); but a complete solution demands improvements in the effectiveness of instruction and in the appropriateness of course content. Also, while the general quality of mathematics teaching, like the general quality of teaching in any field, is probably in need of improvement, one can cite many examples of excellent classroom technique and exciting—even inspired—mathematical instruction in the American schools.

From the point of view of the Commission, it is not important to determine the precise reasons for public concern about the program of secondary school mathematical instruction, or to support or refute such reasons. It is important to recognize that such a general concern exists, and that the existence of this concern strongly supports the thesis that the time is ripe for the improvement of the high school mathematics program. But improvement should not be conceived merely as a by-product of the post-Sputnik panic. Awareness of the problem, the appointment of the Commission, and the beginning of its study all antedate the Sputniks. The dramatic Soviet achievements serve, helpfully, to create greater public concern and greater awareness and acceptance of what must be done. It is imperative that hasty improvisations be avoided, and that action be based on sound judgment and well-reasoned arguments. As in most concerns of importance, one should beware of the pat answer. Individual differences among human beings are the rule; these differences can be neglected only with peril.

The national need for mathematical manpower

Finally, the fast-growing national need for people skilled in various branches of mathematics provides a compelling reason why an improved mathematics curriculum for college preparation is of the utmost importance. Americans traditionally have devoted themselves to practical rather than to theoretical knowledge. The names of certain American inventors are household words — Whitney, Morse, McCormick, Edison, and others; the

names of great American theoretical scientists, except for Einstein (an American by adoption), are not. Over a century ago, de Tocqueville observed, "Hardly anyone in the United States devotes himself to the theoretical and abstract portion of human knowledge." This same imbalance between practical and theoretical preoccupations remains a serious problem to the United States. The Commission does not wish to support its recommendations by playing on fears, but it is an incontrovertible fact that America does have an undersupply of men and women adequately trained in mathematics and science. For example, a committee of the American Association for the Advancement of Science tells us that, "The progress of basic science does not appear to be keeping pace with the development of applied science." Neither basic nor applied science can develop as they ought without adequate mathematical foundations.

The demand for mathematically trained men and women comes not only from science and engineering, as in the past; it now comes from business, industry, and government as well. Within two decades the employment pattern of persons with mathematical training has changed completely. Twenty years ago, college or secondary school teaching absorbed all but a handful of persons holding master's or doctor's degrees in mathematics; there were no other jobs. Today the applications of mathematics in electronics, in the design and use of computing machinery, in industrial research, in automation, and in a dozen other areas have opened up new opportunities and created new demands. Mathematical journals and even daily newspapers carry "help wanted" advertisements for the mathematically trained; qualified teachers are being attracted from the classroom to the laboratory, plant, or office.

One aspect of this startling demand for mathematical manpower is evidenced in a statement made by George E. Forsythe, professor of mathematics at Stanford University, in a lecture presented before the Mathematical Association of America in Cambridge, Massachusetts, on August 30, 1958:[3]

"There seem to be over 3,000 automatic digital computers now installed in the United States, with more on the way. As a rough estimate, each automatic computer needs to have 10 attendants who serve it as mathematicians – programmers, coders, analysts, supervisors, and so forth. The resulting requirement for 30,000 computer mathematicians should be compared with the combined membership of the American Mathematical Society, Mathematical Association of America, Society for Industrial and Applied Mathematics, Association for Computing Machinery, Institute of Mathematical Statistics, and American Statistical Association—under 20,000 persons. While some makeshift arrangements are possible, the disparity in numbers is creating the unprecedented demand . . . for the new A.B. in mathematics."

This pressing national need for what we may call "mathematical manpower" cannot be met merely by training more young men and women in an outmoded curriculum. One well-trained high school graduate may make a more effective contribution to the national manpower need than a half-dozen who are poorly trained.

[3] "The role of numerical analysis in an undergraduate program."

Summary of the case for revision

This then is the case for curricular revision: Mathematics is a dynamic subject, characterized in recent years by such impressive growth and such extensive new applications that these have far outrun the curriculum. Moreover, the traditional curriculum fails to reflect adequately the spirit of contemporary mathematics, which seeks to study all possible patterns recognizable by the mind, and by so striving has tremendously increased the power of mathematics as a tool of modern life. Nor does the traditional curriculum give proper emphasis to the fact that the developments and applications of mathematics have always been not only important but indispensable to human progress.

In order that the school and college curricula meet the needs of mathematics itself and of its applications, there must be a change. A new program, oriented to the needs of the second half of the twentieth century and based on a dynamic conception of mathematics, is required. The national need for mathematical manpower, and a general feeling of dissatisfaction with the present state of affairs, support the early introduction of such a new curriculum.

The necessity for a thorough revision of the program in mathematics (as a basic element of the program in science education) is impressively summarized in the following quotation:

"First, the crisis in our science education is not an invention of the newspapers, or scientists, or the Pentagon. It is a real crisis.

"Second, the U.S.S.R. is not the 'cause' of the crisis. The cause of the crisis is our breath-taking movement into a new technological era. The U.S.S.R. has served as a rude stimulus to awaken us to that reality.

"The heart of the matter is that we are moving with headlong speed into a new phase in man's long struggle to control his environment, a phase beside which the industrial revolution may appear as a modest alteration of human affairs. Nuclear energy, exploration of outer space, revolutionary studies of brain functioning, important new work on the living cell – all point to changes in our lives so startling as to test to the utmost our adaptive capacities. . . .

"The immediate implications for education may be briefly stated. We need an ample supply of high caliber scientists, mathematicians, and engineers. . . . We need quality and we need it in considerable quantity."[5]

[4]Cf. W. W. Sawyer, *op. cit.*, p. 12.

[5]*The Pursuit of Excellence; Education and the Future of America* (Special Studies Project Report V, Rockefeller Brothers Fund), (Garden City, N.Y.: Doubleday and Co., Inc., 1958), pp. 27-28.

2. Secondary education: the Commission's premises

The Commission was appointed for the avowed purpose of improving the program of college preparatory mathematics in the secondary schools. This goal has been before its members at all times. The Commission has no authority to enforce changes; its role is to recommend and to suggest. By these means, it hopes to influence, but not dictate, the immediate future of the college preparatory mathematics curriculum.

In curriculum building, attention must be paid to the purposes of the schools, their place in society, and their programs; to the nature of the pupils, their maturity, their interests, their objectives; and to the findings of psychology as to how human beings grow, develop, mature, and learn. At this point, therefore, it may be appropriate for us to indicate the general premises within which our recommendations for a new curriculum are made.

The role of the secondary school

The members of the Commission recognize that there rests on the American secondary school an obligation to serve "all the children of all the people." We do not believe, however, that this means that the entire school population should be required to take exactly the same program. While we hold that some form of mathematical instruction is needed by every secondary school student, we do not believe that everyone should study the curriculum set forth in this report.

Our program is a college preparatory program, designed for students who can profit from it. We believe the secondary school must meet the needs of this group, as well as the needs of all other groups. *Every* student should have what is for him a challenging and rewarding intellectual experience, marked by sound intellectual content, by appropriate personal and social development, and by preparation for everyday living. All students need not be taught at the same pace, in the same order, or to the same extent, or with the same emphasis.

Mathematics in general education

In many secondary schools, only a minority of the school population proceeds to college. The Commission realizes that secondary schools must serve the needs of those students who are not bound for college. Many

aspects of the Commission's program can be adapted, in terms of general education, to this group of students, though this adaptation is a task that the Commission must perforce leave to other hands.

Instruction in mathematics designed to meet the needs of secondary school students for general education should aim to teach the student the basic mathematical ideas and concepts that every citizen needs to know, and to explain the essential character of mathematics—how it is used to explore and describe physical reality, and how it is used to contribute through its aesthetic values to one's personal intellectual satisfaction. More specifically, some objectives of mathematics in general education are:

1. An understanding of, and competence in, the processes of arithmetic and the use of formulas in elementary algebra. A basic knowledge of graphical methods and simple statistics is also important.

2. An understanding of the general properties of geometrical figures and the relationships among them.

3. An understanding of the deductive method as a method of thought. This includes the ideas of axioms, rules of inference, and methods of proof.

4. An understanding of mathematics as a continuing creative endeavor with aesthetic values similar to those found in art and music. In particular, it should be made clear that mathematics is a living subject, not one that has long since been embalmed in textbooks.

Mathematics for the "college-capable"

There are many secondary schools—usually city or suburban public schools or independent schools—from which a majority of the graduates proceeds to higher education. Whether this group is small or large, in any school it is an important group deserving the most careful consideration. From the college-trained will come the nation's scientists, engineers, physicians, lawyers, teachers, writers, artists, ministers, and other professional men and women—and probably the statesmen who will help shape the world's future.

It is a mistake, in the American educational system, to think in terms of the "college-bound." Financial or other external considerations too often affect the question of who is bound for college. As a policy in education, the nation cannot afford to do less than attempt to recognize and prepare appropriately every student who is capable of college work. The Commission prefers to call such students the "college-capable." Toward this group as a whole, the schools have a major responsibility.

The program set forth in this report is designed to meet the needs in mathematics of college-capable students in the second half of the twentieth century. The Commission believes that bright, young minds will find the material exciting and challenging, demanding and rewarding.

If the Commission is correct in this belief, a task of first importance faces the home and the school: to see that students of college potential tackle this program (or one of similar caliber) for *at least* three years. The shocking inadequacy of the traditional notion that "one year of algebra and one year of geometry" is sufficient college preparation in mathematics should be apparent to all. The handwriting is not only on the wall; it is all over the nation.

Moreover, it should be emphasized that the recommendation of three years of mathematics for the college-capable is minimal. Indeed, the Commission goes much further: it believes that school counselors, teachers, and parents have a duty to see that as many as possible of the college-capable should study mathematics in high school for four years. The most talented should attempt the Advanced Placement Program (described on p. 15) if it can be made available.

These convictions of the Commission are generally supported by a statement recently appearing in one of the Rockefeller Reports and based partly on a study by Dr. James B. Conant and partly on the findings of a conference sponsored by the National Education Association:

"In addition to the general education prescribed for all—four years of English, three to four years of social studies, one year of mathematics, and one year of science—the academically talented student should have two to three additional years of science, three additional years of mathematics, and at least three years of a foreign language. For certain students, the study of a second foreign language, for at least three years, might replace the fourth year of mathematics and the third year of science."[1]

There are cogent reasons for urging that the college-capable student should study mathematics for four years. Young people in high school do not always know what careers they will follow. What they should know beyond all doubt is that lack of mathematical preparation closes many doors — not only doors that open the way to engineering and the natural sciences, but also newer doors that lead to important areas of the social sciences, biological sciences, business, and industry. Many a college professor can testify to the frustrations of graduate students who face formidable roadblocks in sociology, economics, biology, and psychology — roadblocks caused by inadequate high school preparation in mathematics. Every indication points to wider and even more striking uses of mathematics in the immediate future. It is not too much to say that many of the college-capable will find their mathematical needs bounded only by the limits of their talent.

The Commission is aware of the importance of breadth in the high school curriculum. Indeed, all of its members would deplore specialization at the expense of sound, liberal training. But certain subjects occupy special levels in the scale of liberal knowledge. Mathematics (like a foreign language) comes easier to the young. The study of some subjects may be postponed for a while without serious loss; indeed, age makes their appreciation much easier. The study of mathematics is different—for the young, it is now or never. The implications of this fact must be faced squarely. For experience proves that high school is the place where most of our great scientists and mathematicians first acquired the interest that spurred them on to high achievement.

It would be most unfortunate to attempt to justify the four-year study of mathematics solely as preparation for a wide and ever-increasing range of applications. Mathematics is eminently worthy of study in its own right: it is a vital and shining example of mankind's creativity, one of the great cultural achievements of the ages. Few subjects

[1] *The Pursuit of Excellence; Education and the Future of America* (Special Studies Project Report V, Rockefeller Brothers Fund), (Garden City, N. Y.: Doubleday and Co., Inc., 1958), p. 27.

can rival its ability to stimulate the inquiring mind. Every educated man and woman should have an opportunity to know at first hand something of the intellectual excitement and deep satisfaction that mathematics can offer.

Two important points: Two of the Commission's premises regarding mathematics for the college-capable should be made clear. First, students studying college preparatory mathematics should, in our opinion, be taught in groups with similar interests and similar intellectual abilities. We believe that instruction so given increases the challenge to the student and the likelihood of his developing his talents and ability to the full.

The injustice done to capable students by failure to use ability grouping is well pointed out in the Rockefeller Report: "Because many educators reject the idea of grouping by ability, the ablest students are often exposed to educational programs whose content is too thin and whose pace is too slow to challenge their abilities."[2]

In their programs of physical education, schools commonly find no difficulty in reconciling a program of physical training for all with provision of more intensive activity for students with sufficient ability to make school "varsity" or "junior varsity" teams. We see no reason why analogous considerations should not be considered valid in intellectual pursuits as well.

Second, college-capable students who follow the recommendations of this report will not be exposed to certain so-called "practical" courses such as consumer mathematics, installment buying, principles of insurance, and so forth. The Commission does not regret this fact. For it believes that these students will ordinarily develop sufficient mathematical power to acquire such information independently if they need it. One must not assume that an individual's knowledge consists solely of what he has been specifically taught in school.

[2] *Ibid.*, pp. 22-23.

Determining course content

The Commission believes that it is not the responsibility of the mathematicians and teachers of mathematics alone to solve all the problems relating to the mathematical instruction of any individual or group. The motivation, vocational objectives, interest, and ability of students all enter into the picture. Hence, the advice of experts in educational psychology is likely to be helpful in many respects.

We submit, however, that no decision with respect to the appropriateness of including some particular mathematical topic in a curriculum ought to be made without considering the opinions of professional mathematicians and teachers of mathematics. They must decide whether subject matter is obsolete, obsolescent, important, significant, educationally meaningful, likely to be increasingly important, and so on. Just as philosophers and psychologists must be heard as to aims, objectives, learning theory, grade placement of material, methodology, and the like, so the mathematicians and teachers must be heard as to the mathematical content of a program.

It is such advice that the Commission has undertaken to give. Our recommendations are based on the needs of mathematics, on its uses, and on the anticipated needs of the

students and of society, insofar as we can determine them.

The Commission believes also that content in mathematics or any other subject, for that matter, must be appropriate to the level of maturity of the student. Otherwise there can be no meaningful learning. The number of questions that can be asked with respect to the "readiness" of pupils of different levels of intellectual, physical, and emotional maturity to deal with particular concepts in mathematics is exceedingly great. Even for traditional subject matter, psychological research has barely scratched the surface. Since we are suggesting the introduction of some new material, the problem is compounded.

In these circumstances, the Commission has taken a middle course between detailed experimental studies and pronouncements based on a priori judgments. We decided to subject ideas that seemed valid or desirable to at least limited tests of their practicality. We have convened several writing groups, composed of Commission members and other teachers of mathematics, to write instructional units dealing with new materials, or new presentations of older materials. These units have been taught experimentally with satisfactory results, indicating the realistic and practical nature of this program as the beginning of the solution of an urgent problem, work toward which cannot be delayed.

The attitude of the Commission is reflected independently in sentiments expressed in an article by Dael Wolfle:

"There are . . . educational changes that lend themselves to experimental study, but many of the current efforts to improve the teaching of science and mathematics do not. . . . There is still room for the exercise of good judgment in attempting to bring about social and educational improvements. If we have what we think is a good song to breathe into the air, let us go ahead and breathe it, without trying to fool ourselves by pretending to be carrying out an experiment."[3]

The place of calculus

The Commission takes the position, held generally in the United States at present, that calculus is a college-level subject. A reasonable immediate goal for most high schools is a strong college-preparatory mathematics curriculum that will have students ready to begin calculus when they enter college. Such is the curriculum described in this report. At the same time, however, the Commission recommends that well-staffed schools offer their ablest students a year of college-level calculus and analytic geometry as recommended in the Advanced Placement Program.[4] It is essential, though, that such a year be firmly based on a full pre-calculus program, completed early by some form of acceleration. In the long run, improvements in the curriculum (beginning with the first grade) and in teachers' qualifications may eventually make it possible to move such a calculus course into the normal program for grade 12 of most schools.

A few schools now teach calculus effectively in their regular programs. But schools that can do this are unusual at the present

[3] "The Fetish of Experiment," *Science*, vol. 125, no. 3,240 (Feb. 1957), p. 177.

[4] The Program is described on p. 15. An encouraging fact is the rapid increase in the number of schools offering Advanced Placement courses in mathematics. In 1956, 45 schools had 386 candidates for the Advanced Placement Examination in mathematics; in 1959, according to an advance estimate, over 360 schools will have over 3,200 candidates.

time. In regarding calculus as a college-level subject, the Commission does not want to discourage those with exceptional facilities from attempting exceptional tasks. Rather, it warns against premature general acceptance of a curricular responsibility for which all too few schools are now adequately prepared.

The Commission cannot recommend the practice of exposing college preparatory students to formal calculus for a short time at the end of the twelfth grade. Such anticipation tends to breed overconfidence and blunt the exciting impact of a thorough presentation. The Commission's program for grade 12 includes some concepts close to calculus—limit, difference quotient, and slope of a curve. But it intends that these should be taught as informal preparation for calculus, rather than as preliminary calculus employing formal language and notation (such as "derivative" and "dy/dx").

The Advanced Placement Program

Under the Advanced Placement Program, which is sponsored by the College Entrance Examination Board, high schools are encouraged to offer to their ablest students college-level courses in the senior year. As has already been suggested, the objectives and philosophy of the Program are in complete accord with those of the Commission. The mathematics course recommended by the Advanced Placement Program is a college-level course in calculus and analytic geometry for twelfth-grade students who have already completed a full high school program, such as the one set forth in this report. The recommendations of the Commission and those of the Advanced Placement Program are therefore quite consistent.

Many larger schools will be interested both in reorganizing their college preparatory work along the lines recommended by the Commission, and also in organizing Advanced Placement courses for their ablest students. The Commission strongly endorses such a plan, provided that students admitted to the Advanced Placement course will have completed, prior to their senior year, the program set forth in the following chapter at least to the end of the course entitled "Elementary Functions."

The program and the teacher

The role of the teacher is vital: curricular change must be accompanied by effective, meaningful teaching, directed toward the development of mathematical power and understanding. Hence, the Commission has made numerous suggestions and recommendations with respect to teacher education, both for the pre-service preparation of teachers for improved programs, and for the in-service training that may produce similar competence. These are set forth in Chapter 5 of this report. We commend them to the attention of school boards, superintendents, teachers' associations, colleges, and universities.

The Commission regards as one of the most important advantages of its approach to curricular revision the fact that it does not demand an "all or none" approach. It does not propose closing the books on the old curriculum on a certain date and beginning a totally new program when the schools next open, but rather that teachers may in-

troduce modifications as the way opens, and as their knowledge and experience permit them to do so.

The interest of teachers in its work and their reactions to its progress reports have confirmed the Commission in its belief that many high school teachers of mathematics are aware of the need to improve the secondary school curriculum, and are receptive to the introduction of new ideas, provided only that they are given suitable assistance in comprehending them and in adapting them to classroom instruction.

Summary

The secondary school has an obligation to serve the needs of all young people, including the college-capable. In providing for this latter group, cognizance must be taken of the high potential for worthwhile academic achievement. The times demand a program in mathematics that exploits this potential—a program that provides for the full development of the individual and meets the needs of the nation. Schools must provide such programs. Parents, teachers, and counselors must face squarely the implied responsibilities. For, as many as possible of the college-capable must be urged to study challenging mathematics for four years in high school; none of them should study mathematics for less than three years, and the most gifted should accelerate their studies in mathematics so as to undertake the Advanced Placement Program. The Commission believes that such studies are likely to be most effective under some system of ability grouping.

It follows that development of programs in high school mathematics suitable for our college-capable students in the last half of the twentieth century is an urgent, present need. As a first approximation to such a program, the Commission decided to make recommendations based on the judgment of experienced mathematicians and teachers, supplemented by some experimentation. The resulting program aims to infuse the spirit of contemporary mathematics into some traditional topics, and to include some new topics important to mathematicians and accessible to high school students. Calculus is regarded as a college-level subject, available in secondary schools through the Advanced Placement Program.

The major demands of any curricular revision fall on the shoulders of the teachers. In appreciation of this fact, the Commission advocates a new program that grows out of the old in a manner that makes reasonable demands on teachers, allowing them to implement the recommendations gradually as their training permits.

3. Recommendation: the Commission's program

In the present chapter, the Commission sets forth recommendations that we believe will achieve the objective of improving the program of college preparatory mathematics in secondary schools. Before discussing these recommendations in detail, we shall make some general comments about their nature.

While the direction of the proposed changes is oriented to the future, the changes themselves are not radical. No attempt has been made to uproot the traditional curriculum. The Commission has tried to have new ideas and new subject matter grow out of the old through reasonable modifications and additions. The result is a revision that the Commission believes can begin at once gradually and move forward rapidly. This revision is designed to meet imperative present needs; the demands of the more distant future in college preparatory mathematics will require further adjustments. But, for the present, the Commission feels that its recommendations offer a goal that is both challenging and attainable.

Furthermore, the Commission's recommendations as to mathematical content have been kept flexible, so that they may be adapted to the varying needs of many different types of schools or groups of students. It is possible to organize the proposed subject matter in a variety of ways. Writers of textbooks will undoubtedly develop the program in accordance with their own understanding of learning theory and their own philosophies of secondary education.

This flexibility of the recommended program includes its manner of presentation. Members of the Commission would decry an authoritarian approach to method and practice, but a teacher who believes that such an approach is most effective may present this material in the same way that he has, presumably, taught the traditional content. Most if not all of the Commission members would prefer to see a developmental approach, which would encourage the student to discover as much of the mathematical subject matter for himself as his ability and the time available (for this is a time-consuming method) will permit.

Although most of the Commission would like to see the interrelations among the various topics of the program pointed out and stressed, no one of them would say that there is one and only one way of accomplishing this. Questions about methods of teaching and patterns of organization; questions as to whether algebra, geometry, trigonometry,

coordinate geometry, and other areas should be presented as separate courses or units, or should constitute a single, integrated fabric; questions about grade placement of material — such questions as these have no ideal answers, valid at all times and in all schools. They are best left to teachers and supervisors, authors and editors. The Commission's hope is that many diverse proposals will be formulated and will compete in the market place of ideas as well as in the textbook market.

In the material that follows, the mathematics supposed to have been covered prior to the program recommended for grades 9 through 12 — in grades 7 and 8, whether in junior high or elementary school — is presented first. Second, the mathematics recommended by the Commission for grades 9, 10, and 11 is discussed. Recommendations for the revision of content and the Commission's reasons for making them are both set forth. Although this statement is presented in terms of traditional subject matter—algebra, geometry, and trigonometry—the form of its presentation does not necessarily represent the order in which the Commission believes these topics should be organized or taught. Third, the program recommended for the twelfth grade is described.

In the next chapter, the Commission outlines a suggested sequence of topics that incorporates its recommendations for the mathematics of grades 9 through 12. The purpose of this sequence is to exemplify one possible course of study that reflects the spirit of the Commission's proposals. Both the proposals of this chapter and the sequence of the next are amplified, clarified, and elucidated in the volume of appendices to this report.

Prerequisite mathematics

The Commission's program is necessarily predicated on certain learnings expected of students before they begin the program. There are in the United States a number of different arrangements of school organization, commonly known as 8-4, 6-3-3, 6-6, and 7-5, where the numbers indicate, respectively, years of study in each division of an elementary and secondary school system. Presumably, regardless of school organization, the mathematics studied year by year should be practically the same under all plans. This is not the case, however. Textbooks written for the seventh and eighth year of study in an eight-year elementary program (8-4) stress arithmetic more, and informal geometry much less, than books written explicitly for a junior high school program (6-3-3).

The Commission recognizes the need for a careful study of the mathematics program of grades 7 and 8 (and, indeed, of the earlier grades as well). But this is not the task of the Commission. To fill this gap, it must rely on other groups such as the School Mathematics Study Group, the University of Maryland Mathematics Project, and the Curriculum Committees of the National Council of Teachers of Mathematics.

This much can be said: regardless of the form of school organization, in order to give students in grades 7 and 8 the type of mathematics study that will form a proper foundation for the Commission's program, the following subject matter is regarded as essential. The Commission is convinced that it can be mastered by all college-capable students during grades 7 and 8. The better students can cover this work in one to one-and-

a-half years and move on to the Commission's program during the eighth school year.

Arithmetic

Fundamental operations and numeration: Mastery of the four fundamental operations with whole numbers and fractions, written in decimal notation and in the common notation used for fractions. This includes skill in the operations at adult level (i.e., adequate for ordinary life situations) and an understanding of the rationale of the computational processes. Understanding of a place system of writing numbers, with use of binary notation (and perhaps other bases) to reinforce decimal notation. Ability to handle very large numbers (greater than 1,000,000) and very small numbers (less than one ten-thousandth). The meaning and use of an arithmetic mean. In addition, a knowledge of square root and the ability to find approximate values of square roots of whole numbers is desirable. (The process of division and averaging the divisor and quotient – Newton's method–is suggested.)

Ratio: Understanding of ratio as used in comparing sizes of quantities of like kind, in proportions, and in making scale drawings. Per cent as an application of ratio. Understanding of the language of per cent (rate), percentage, and base. In particular the ability to find any one of these three designated numbers, given the other two. Ability to treat with confidence per cents less than 1 and greater than 100. Applications of per cent to business practices, interest, discount, and budgets should be given moderate treatment.

Geometry

Measurement: The ability to operate with and transform the several systems of measure, including the metric system of length, area, volume, and weight. Geometric measurements, including length of a line segment, perimeter of a polygon, and circumference of a circle, areas of regions enclosed by polygons and circles, surface areas of solids, volumes of solids, measure of angles (by degrees). The use of a ruler and protractor. The student should know the difference between the process of measuring and the measure of the quantity. Ability to apply measurement to practical situations. Use of measurement in drawing to scale and finding lengths indirectly.

Relationships among geometric elements: These include the concepts of parallel, perpendicular, intersecting, and oblique lines (in a plane and in a space); acute, right, obtuse, complementary, supplementary, and vertical angles; scalene, isosceles, and equilateral triangles; right triangles and the Pythagorean relation; sum of the interior angles of a triangle. The use of instruments in constructing figures; ideas of symmetry about a point and a line.

Algebra and statistics

Graphs and formulas: Use of line segments and areas to represent numbers. Reading and construction of bar graphs, line graphs, pictograms, circle graphs, and continuous line graphs. Meaning of scale. Formulas for perimeters, areas, volumes, and per cents–introduced as generalizations as these concepts are studied. Use of symbols in formulas as placeholders for numerals arising in meas-

urement. Simple expressions and sentences involving "variables."

Mathematics for grades 9, 10, and 11 (Elementary and Intermediate Mathematics)

In this discussion and throughout this report, the work of grades 9 and 10 will often be designated as Elementary Mathematics I and II, and the work of grade 11 as Intermediate Mathematics. This work ordinarily will be covered in grades 9 through 11, but schools may, of course, begin the work earlier or extend it later. Indeed, it is precisely because of the Commission's desire to provide for as much flexibility in organization as possible that we have adopted such generalized descriptive terms as Elementary Mathematics, Intermediate Mathematics, and Advanced Mathematics, adaptable to various grade designations.

Algebra

The Commission recommends increased attention to algebra as a part of the secondary school curriculum, but couples it with an equally earnest recommendation that the point of view from which the material is presented be that of contemporary mathematics, as explained in the opening chapter (p. 2). The goal of instruction in algebra should not be thought of exclusively or even largely as the development of manipulative skills. Rather instruction should be oriented toward the development and understanding of the properties of a number field. At this point it is probably well to digress and present two points to guard against possible misunderstanding.

First, in stating that the material of algebra should be presented from the point of view of contemporary mathematics and that instruction should be oriented toward the properties of a number field, the Commission is *not* advocating the outright presentation of elementary algebra from an abstract point of view. Nor does it advocate the teaching of abstraction before concrete or intuitive notions have been established as a point of departure.

Development of generalized abstract concepts is difficult, too difficult to be a point of departure for the beginner, but nevertheless it is absolutely necessary if any true understanding of mathematics is to be attained. It cannot be used as a starting point until the proper intuitive foundation has been laid and the students are ready for it. Many of the "laws" of algebra (the axioms of a number field) are easier to understand and much more useful than the arbitrary "rules" usually found in elementary textbooks. These laws should be taught and used—at an appropriate stage in the instructional process. Such ideas are more fully set forth and illustrated in the first appendix to this report, "An introduction to algebra" (see separate *Appendices* volume).

Second, the Commission fully realizes the necessity of teaching appropriate manipulative skills. Its members are as concerned as anyone that students entering college mathematics courses should be able to solve quadratic equations and systems of linear equations, perform the fundamental operations upon polynomials and rational fractions, and deal routinely with the other simple algebraic operations necessary for comfortable progress in a course in calculus and analytic geometry. But the development of these ma-

nipulative skills should not be the most important nor the only goal of instruction in algebra; many college professors would no doubt share the view of one who stated recently that he would gladly forego such skills if he could be assured instead of a genuine understanding of deductive reasoning.

It is not, however, an alternative of *either* skill *or* understanding that confronts teachers. *Both* skills *and* concepts are essential. The Commission is not alone in insisting that the development of manipulative skill should not be the primary goal of instruction in algebra. Nor is this a novel idea. The National Committee on Mathematical Requirements, writing in 1923, stated: "The excessive emphasis now commonly placed on manipulation is one of the main obstacles to intelligent progress. . . . Drill in algebraic manipulation should be limited to those processes and to the degree of complexity required for a thorough understanding of principles and for probable applications either in common life or in subsequent courses which a substantial proportion of the pupils will take. It must be conceived throughout as a means to an end, not as an end in itself. Within these limits, skill in algebraic manipulation is important, and drill in this subject should be extended far enough to enable students to carry out the essential processes accurately and expeditiously." [Italics omitted.][1]

This statement is as valid today as it was a generation ago when first written. It applies with special significance to the study of college preparatory mathematics. Strong skills are surely needed; but they must be based on understanding and not merely on rote memorization. Once meaning has been achieved, then drill should be provided to establish skills—skills that can be performed, as Whitehead says, "without thinking." In this way, the mind is liberated to grapple with new ideas.

The new with the old: In making its recommendations for increased emphasis upon algebra and for instruction oriented toward a more contemporary point of view, the Commission is influenced by the fact that algebra is a subject that has been largely transformed by the mathematical research of the past quarter-century. This transformation has been brought about through the systematic axiomatic development of algebra or, more accurately, of algebras, and the light thus thrown on the study of mathematical structures. Such structures, or patterns, have been thrown into sharp relief. The Commission is influenced too by the fact that many recent applications of mathematics to areas hitherto regarded essentially as non-mathematical are algebraic in character.

In the proposed program, the mechanics or formal manipulations in algebra are the same as hitherto taught, and the subject matter is largely the same. The difference is principally in concept, in terminology, in some symbolism, and in the introduction of a rather large segment of new work dealing with inequalities treated both algebraically and graphically. Solution sets of inequalities involving two variables are also studied.

The new emphasis in the study of algebra is upon the understanding of the fundamental ideas and concepts of the subject, such as the nature of number systems and the basic laws for addition and multiplication (com-

[1]National Committee on Mathematical Requirements, *The Reorganization of Mathematics in Secondary Education* (n.p.: The Mathematical Association of America, 1923), p. 11.

mutative, associative, distributive). The application of these laws in various number systems, with emphasis on the generality of the laws, the meanings of conditional equations and identities and inequalities, is stressed. The nature of a function – in particular, the linear, quadratic, exponential, and logarithmic functions–is also discussed.

As an example of the use of a fundamental concept in illuminating algebra, the distributive law may be cited. This law is the basic idea behind much mental arithmetic, the use of parentheses, factoring, multiplication of polynomials, and the manipulation of fractions. If it is understood, most of the special methods of handling these topics can be eliminated.

Deductive reasoning in algebra: One way to foster an emphasis upon understanding and meaning in the teaching of algebra is through the introduction of instruction in deductive reasoning. The Commission is firmly of the opinion that deductive reasoning should be taught in all courses in school mathematics and not in geometry courses alone.[2]

A student who understands the nature of a subject is more likely to be able to solve problems that present an element of novelty than one who lacks this understanding. The ability to solve such problems involves more than merely the blind application of rules or techniques to typical problems preclassified as to form.

Moreover, in their ability to perform algebraic manipulations, students who study algebra from the point of view advocated in this report should be the equal of students who study more traditional curricula. When algebra (or any subject, for that matter) is studied or taught with an eye to the fundamental nature of the subject, this more general point of view tends to make the whole subject more understandable. Diverse bits of information support one another and "hang together" so to speak, thereby promoting understanding and assisting memory.

For example, one sometimes reads: "If the discriminant of a quadratic equation is a perfect square, then the roots of the equation are rational." Yet the equation

$$2x^2 + 2\sqrt{6}\,x + 1 = 0$$

has a discriminant equal to 16, and irrational roots. Wherein lies the discrepancy? A student who approaches algebra from the contemporary point of view is accustomed to thinking in terms of the set of numbers admissible in a particular problem. The quoted statement holds only for quadratic equations with rational coefficients: it fails for the given example because $2\sqrt{6}$ is not a member of the set of rationals.

Geometry

Objectives of geometric study: The inclusion of geometry in the high school curriculum has three main objectives.

The first objective is the acquisition of information about geometric figures in the plane and in space. Since geometry originated as, and still essentially is, a mathematical model of the physical world, the student needs to know the facts of geometry if he is to be able to deal effectively with the world about him. This knowledge is important for the everyday citizen and essential for the prospective scientist. Moreover, little prog-

[2] As a specific example of how instruction in deductive reasoning may be introduced into algebra, even very early in the course, the Commission prepared a classroom booklet entitled, *Informal deduction in algebra.* (This booklet is now out of print.)

ress can be made in trigonometry or calculus without an understanding of geometric facts.

The second objective is the development of an understanding of the deductive method as a way of thinking, and a reasonable skill in applying this method to mathematical situations. For historical reasons, the deductive method has been emphasized in geometry but not elsewhere in high school mathematics. However, it is now possible and desirable to use the deductive method in all mathematical subjects, and consequently the time devoted to it in geometry can be somewhat reduced.

Not all reasoning is syllogistic or deductive. Training in mathematics based on deductive logic does not necessarily lead to an increased ability to argue logically in situations where insufficient data exist, and where strong emotions are present. It is a disservice to the student and to mathematics for geometry to be presented as though its study would enable a student to solve a substantial number of his life problems by syllogistic and deductive reasoning.

Deductive methods are taught primarily to enable the pupil to learn mathematics. Mathematics, and consequently deductive methods, can be applied to life only in those life-situations that are capable of accurate transformation into mathematical models. These situations, though of tremendous importance, are far from frequent in the everyday lives of high school students.

A third important objective of the geometry course is the provision of opportunities for original and creative thinking by students. The material of elementary geometry affords an uncommonly good opportunity for students to think along lines that are original for them. Its elements are sufficiently simple to be grasped readily, and its consequences sufficiently complex to challenge students of varying abilities, no matter how high. Therefore a large part of the course should be devoted to original exercises involving, if possible, both the discovery of relationships and their proofs.

Defects and possible remedies: Geometry must continue to play a large role in secondary school mathematics. The Commission recommends, however, that the character of the course in geometry be drastically changed. Recent developments in geometric thinking have disclosed grave faults in the logical structure of Euclid, thus calling attention to the need for a modification in the traditional approach to high school geometry.

There are essentially three defects in the Euclidean development of geometry that make it unsuitable as the basis for modern high school instruction. Since these are discussed at length in Appendices 10 ("Some reasons for modifying the traditional treatment of geometry") and 18 ("Order relations in plane geometry"), they receive only brief mention here. The first defect is Euclid's failure to formulate explicitly the axiomatic basis on which the congruence theorems rest. Although there is in the *Elements* an extensive argument in support of the superposition of line segments, the less obvious matter of justifying the superposition of angles is completely neglected. A remedy for this defect is the outright assumption of the congruence theorems.

The second defect is the difficulty introduced into some phases of the development because of the lack of an adequate algebra. Today, of course, these difficulties can be avoided by the use of coordinate geometry based on a one-to-one correspondence be-

tween ordered pairs of real numbers and points in the plane.

The third defect is the failure of Euclid to recognize the necessity of making formal assumptions concerning "betweenness," an omission recognized by Gauss 125 years ago and first corrected by Pasch in 1882. For any complete logical formulation of geometry, there must be postulates concerning the *order* of points on a line and the matter of regions in the plane, neither of which is mentioned in Euclid's *Elements*. A discussion of order relations that illustrates the kind of treatment needed to make Euclid logically rigorous is given in Appendix 18 ("Order relations in plane geometry").

What should be done to remedy the situation created by the foregoing logical defects in Euclid's development? One possibility is that of developing a logically unimpeachable treatment of Euclidean geometry suitable for use in secondary schools. Although some distinguished mathematicians, among them D. Hilbert, O. Veblen, G. D. Birkhoff, R. L. Moore, and S. MacLane, have undertaken this task, the Commission feels that at present no presentation suitable for high school exists. The danger is that of boring students with logical subtleties of little or no interest to most of them, and with proofs of theorems that seem quite obvious.

The Commission hopes that mathematicians will continue to strive for the development of a logically sound basis for Euclidean geometry in a form suitable for high school use. However, for the present, it recommends that textbook writers and teachers should feel free to modify the Euclidean development to attain a more incisive and interesting program.

A mathematical science, when put in final form, consists of a set of undefined terms and unproved propositions, in terms of which all other concepts are defined and all other propositions are proved. It is, of course, a mark of mathematical elegance to reduce the number of undefined concepts and simple unproved propositions to a minimum. Moreover, if in an axiomatic treatment of any branch of mathematics, one takes as an assumption a proposition that might have been proved on the basis of the other assumptions, the beauty or elegance of the structure is somewhat marred. One cannot tell whether the proposition is an assumption or a theorem. But there is nothing erroneous about this situation, so long as the assumptions are consistent with one another.

For secondary school purposes it is clearly not necessary that the set of unproved propositions be reduced to the theoretical minimum; it is better to assume propositions which seem so obvious that proving them seems meaningless to the student, even though proofs of them could be effected in terms of the other assumptions. Furthermore, the geometry course need not consist of a single chain of deductions from a set of primitive terms and primitive propositions lasting through the entire course; and it certainly should not be based on an axiomatic treatment like that of Euclid, which without modification involves serious defects.

Specific proposals for geometry

As has already been set forth, it is felt that a substantial introduction to geometry on an intuitive and informal basis should be accomplished in the seventh and eighth grades. A sense of geometric form, a knowledge of simple geometric facts, and skill in simple

geometric constructions are all appropriate achievements for junior high school or upper grade-school pupils. Deductive proofs will contribute little at best to the understanding of these aspects of geometry. In a sense, this intuitive geometry may be thought of as the physical geometry of the space in which we live, rather than as an abstract mathematical system. These remarks should not be taken to imply that deductive thinking is to be considered unimportant in geometry instruction, but merely that it is not an appropriate aim for the introductory course *preceding* the ninth grade. However, it is to be regarded as a major aim of the senior high school program in geometry.

With respect to the latter course in geometry, the Commission has several observations to make.

The number of theorems should be reduced: It has been traditional in geometry courses for students to be held responsible for the formal proofs of a considerable number of theorems arranged in a precise, sequential order. Often the number of such theorems has been substantial. The Commission believes that the number of basic theorems should be drastically reduced, and that the geometry course should consist of several short sequences, rather than a single long sequence of theorems.

It should be emphasized that these basic theorems are not the only theorems that would be proved. All that is being said is that there would be only 10 or 12 propositions whose proofs would be required as part of an inviolable deductive chain. All other propositions would be treated as originals. A student might offer different proofs on different occasions, and order would be relatively unimportant. Of course, no teacher would accept such circular reasoning as basing the proof of two propositions each upon the other.

In making this recommendation for drastically reduced deductive sequences, the Commission is independently in agreement with a report recently issued in England by representatives of examining bodies and teachers' associations. The report recommended that students be held responsible in examinations for only certain "key-theorems," on the ground that otherwise too much teaching time will be sacrificed to learning the proofs of theorems. The key-theorems suggested by the group are only six in number, and the report comments that "The proposed list . . . is not necessarily the best possible selection, but it is of about the right length if there are to be theorems at all."[3]

The sequences of theorems suggested by the Commission are set forth in the following chapter ("Mathematics for grade 10," section III, p. 38; section V, p. 39). The objective of the first sequence, it will be noted, is to reach the Pythagorean Theorem at an early stage.

Coordinate geometry should be introduced: The Commission proposes that some coordinate geometry be introduced early in the geometry course. An ideal time is immediately after the first sequence of theorems ("Mathematics for grade 10," section III, p. 38). The student will then be able to combine the geometric facts of the sequence with his earlier experiences in graphical algebra, and thus have the satisfaction of seeing two rivers of knowledge join to form a mightier

[3]G. B. Jeffery, *School Certificate Mathematics* (London: Cambridge University Press, 1944), p. 3.

mathematical stream. The student will also need coordinate geometry to deal with trigonometry in the manner recommended in this report. But there are deeper reasons for introducing some coordinate geometry at this time. The discovery by Descartes that the position of any point in the plane can be fixed by an ordered pair of real numbers is one of the most momentous in the history of mathematics. The student will be ready to use this idea at the point indicated, and the intellectual opportunity should not be postponed.

It has been mentioned that coordinate geometry avoids the defects in the Euclidean development that stem from Euclid's lack of an adequate algebra. There are other advantages: the student gains new power because he has a general method for studying geometry; and he achieves results that later can be extended to three and more dimensions. Moreover, he is laying firm foundations for the future: college work, from calculus on, employs geometric material in analytic form.

Whitehead has underlined the important role of coordinate geometry in mathematical thought:

"No one can have studied even the elements of elementary geometry without feeling the lack of some guiding method. Every proposition has to be proved by a fresh display of ingenuity; and a science for which this is true lacks the great requisite of scientific thought, namely, method. Now the essential point of coordinate geometry is that for the first time it introduced method. . . . [It] relates together geometry, which started as the science of space, and algebra, which has its origin in the science of number."[4]

Once coordinate geometry has been introduced, the Commission advocates the use of analytic (algebraic) as well as synthetic methods in proving geometric theorems and exercises. Indeed, certain propositions might advantageously be given both analytic and synthetic proofs. Simple examples of propositions amenable to such treatment are: (1) the diagonals of a square bisect each other at right angles; (2) the medians of a triangle are concurrent at a point that divides each median in the ratio 2:1; and (3) the line segment joining the mid-points of two sides of a triangle is parallel to the third side and equal to half of it (these and others are given and commented on in the volume of appendices, Appendix 15, "Theorems having easy analytic proofs").

Place of solid geometry: The geometry course need not and probably should not be sharply divided into two parts on the basis of dimensionality. It is not necessary completely to separate plane and solid geometry. On the contrary, it is both possible and desirable to teach certain material of solid geometry along with the analogous content of plane geometry. For example, the locus of all points equidistant from two fixed points can be considered in two dimensions and then in three dimensions. Similarly, the treatment of the sphere may be coordinated with the treatment of the circle, developing new insights by the comparative study of properties in two and three dimensions.

Time equivalent to about one-third of a semester should be devoted to solid geometry. The objective should be the development of concepts of spatial relations, and of spatial perception. Mensuration should have been covered intuitively earlier; however, lengths, areas, and volumes of standard figures should

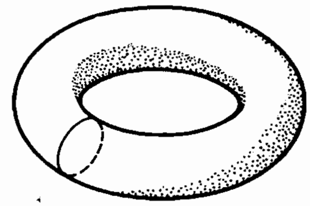
[4]Alfred North Whitehead . . . *Introduction to Mathematics*, (London: Oxford University Press, 1948), pp. 83-84.

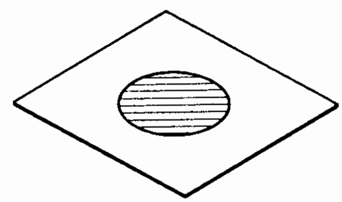
be reviewed. Deductive proofs of mensuration formulas are not in order at this time: such demonstrations are properly a part of integral calculus. Theorems in spherical geometry are particularly suitable material, since they can be compared and contrasted with theorems in plane geometry.

Coverage of solid geometry should include the basic facts about lines, planes, angles, dihedral angles, and spheres. There is neither time to establish all these facts deductively, nor virtue in so doing. A clear understanding of the relationships is essential, however. This can best be attained by an intuitive approach. An outline of the material the Commission considers essential is given in the volume of appendices (Appendix 16, "Outline of a unit in solid and spherical geometry").

Other geometries: It has been known for over a century that Euclidean geometry is not the only possible geometry of congruence. This is an important fact for students to appreciate. Although time will not permit doing much about these "non-Euclidean" geometries, some ideas can be suggested to students by using spherical geometry as a foil. For example, on a sphere the sum of the angles of a triangle always exceeds two right angles, and two triangles are congruent if corresponding angles are equal.

There are still other types of geometry—such as projective geometry and topology—that are not concerned with congruence. Simple pictorial material from these geometries provides intriguing foils to stimulate the imagination. For example, on a torus (the surface of a doughnut) a cut along a circle through the hole leaves the surface still in one piece; but in a plane a cut along any circle severs the plane into two pieces (see accompanying figures).





Summary of the proposed course in geometry: In brief, the program proposed by the Commission envisages an informal and intuitive introduction to geometric ideas, followed by an informal discussion of the nature of deductive reasoning. The course would then take up a short but important sequence of theorems, studied deductively, culminating in the Pythagorean Theorem. In the treatment of this sequence, both the geometric and the logical ideas previously introduced informally would be illustrated, consolidated, and confirmed by more formal study. Specific postulates, including in addition to the usual assumptions the congruence properties of triangles, would be introduced.

With the Pythagorean Theorem and theorems about similar triangles established, it is possible to proceed with coordinate geometry. The essential topics to cover are:

1. Location of points by coordinates
2. Length and slope of a line segment
3. Division of a segment in a given ratio
4. Equation of a line

5. Equation of a circle

(For a discussion of these topics, see Appendix 14, "Introduction to coordinate geometry.")

The remainder of plane geometry can now be developed using both synthetic and analytic methods. Emphasis should be placed on the development of skill in analyzing a situation and constructing a valid proof of either kind. Additional short sequences of theorems are suggested in the next chapter (p. 39, section V, "Mathematics for grade 10").

Since the scope of the analytic method may be unfamiliar to some teachers, a list of important theorems that have simple analytic proofs is presented in the volume of appendices (Appendix 15, "Theorems having easy analytic proofs"). Suitable exercise material can be found in almost any textbook on analytic geometry.

In recommending the introduction of relatively short deductive chains of theorems and a mixture of synthetic geometry with coordinate geometry, the Commission is aware that some teachers may regret the loss of an opportunity to develop an extended sequence of theorems. The Commission does not share the regret, nor do we believe that any important loss to the student is involved. Able students need not lose the opportunity for the appreciation of a complex mathematical structure derived from a set of assumptions. Books in which they can encounter this particular form of mathematical beauty are easily available. Moreover, able students, and others as well, will benefit from the greater power and the increased opportunities for original thinking provided by the new organization of subject matter.

Trigonometry

Trigonometry is the part of school mathematics related most clearly to technical applications. In the past, these had mainly to do with surveying and navigation. Therefore, much attention was paid to methods of solving plane and spherical triangles by logarithmic computation. But that era has passed. Special tables, computing machines, and other equipment have made the logarithmic solution of triangles an almost obsolete tool. Instead, there are more substantial and challenging applications of trigonometry now evident in many areas of science and technology—especially, in statics and dynamics, electromagnetic waves, and vibration problems of all sorts.

Trigonometry must be reorganized to meet these contemporary needs. Computational emphasis should shift from triangles to vectors, and analytic emphasis from identities to functional properties. Of course, some triangles will still be solved (by direct use of the laws of sines and cosines), and simple identities still treated. However, the vital material of the reorganized trigonometry lies in the rectangular and polar description of points, vectors, and complex numbers, and in the addition theorems and periodic character of the circular functions. In particular, the one-to-one correspondence between the set of ordered pairs of real numbers, and the set of points of the plane, or the set of vectors drawn from the origin, or the set of complex numbers, provides a superlative example of the manner in which important mathematical ideas coalesce.

In the sequence outline suggested by the Commission in the following chapter there are four trigonometric units (see p. 43), here

briefly indicated in the following manner:

1. Rudimentary trigonometry of right triangles
2. Trigonometry of x, y, r, θ — coordinates, vectors, complex numbers
3. Cosine and sine laws, addition theorems, identities
4. Circular measure, circular functions and their wave nature.

The first is an optional unit at the end of grade 9, the second and third occur at the end of grade 11 (about a third of a semester), and the fourth forms the final part of the semester course in elementary functions in grade 12. The first unit might be anticipated in grade 8, or combined with unit two in grade 11.

Mathematicians now put much stress on the definition of the circular functions in terms of real numbers. One way of developing this idea is by winding a line around a unit circle. The Commission recommends that this be done in unit four. On the other hand, unit two treats functions of angles (in degrees) — the entities that occur in direct applications to physical vector quantities. Radian measure might be introduced in unit three, but it is not essential at that stage. It is essential at the beginning of unit four as the natural means of transition from the functions of angles used in vector computation to the functions of real numbers used in trigonometric analysis. (As a physicist says, the radian measure of an angle is the ratio of two lengths and so is a pure number.)

Treatment of complex numbers as vectors is a most important mathematical by-product of the reorganized trigonometry. Reinforcing the student's earlier contact with complex numbers in the study of quadratic equations, this trigonometric treatment should round out a good basic knowledge of complex numbers in grade 11. At the end of unit four, an informal discussion of Euler's formula ($e^{ix} = \cos x + i \sin x$) and series expansion serves to relate the exponential and circular functions in striking fashion, and to free the latter from angles without a shadow of doubt.

It is also possible to develop the essentials of trigonometry (included in units two through four) by starting with the circular functions in terms of real numbers and ending with angles, radian measure, and degree measure. Such passage from "pure" to "applied" trigonometry may have greater brevity and mathematical elegance, but it presents a higher level of abstraction to the learner. The Commission's proposed treatment of trigonometry should not be construed as a dogmatic judgment in favor of angles over real numbers. Rather, it reflects the Commission's decisions to stress coordinates, vectors, and complex numbers in grade 11, and functional properties in grade 12.

An introduction to statistical thinking

Just as mathematics deals with situations in which the facts can be determined, it also provides ways to study, understand, and control uncertainty. Many of the newer applications of mathematics use the theories of probability and statistical reasoning. Increasingly, modern science—physics, biology, social science—makes use of probabilistic descriptions of phenomena.

The Commission believes that it is desirable that material in these areas be introduced into the high school curriculum. Statistical thinking is playing more and more of a part in the daily lives of educated men

and women. An introduction to statistical thinking is an important supplement to an introduction to deductive thinking.

This introduction may well begin in the ninth grade or earlier with a unit on descriptive statistics (like that outlined on p. 37, section X, "Mathematics for grade 9"); it would cover numerical data, frequency tables, averages (median, mean), and simple measures of dispersion (range, quartiles). A more formal unit on probability, designed for grade 12, is discussed later in this chapter (p. 31).

Mathematics for grade 12 (Advanced Mathematics)

The subject matter of a fourth high school year is treated separately because this course will not necessarily be pursued by all students who have completed the work of the earlier years. College entrance requirements for students wishing to take nontechnical or nonscientific courses in college frequently do not demand more than Elementary and Intermediate Mathematics. The Advanced Mathematics course will be taken by students whose personal interest in mathematics impels them to elect this course. Because there is a certain element of natural selection about the enrollment, it is possible to suggest content for this course that can properly be described as Advanced (high school) Mathematics.

Many possibilities for topics: It should be noted that the character of the study of mathematics undergoes a change at about the twelfth-grade level: the sequential aspect of the study of mathematics begins to break down. In the early grades, one can be reasonably certain about many parts of the sequence—arithmetic before algebra, algebra before trigonometry, and so forth. But by about the middle of the twelfth grade the student's mathematical journey has brought him to a point from which he can proceed in many directions, all of them inviting. Consequently, there are many topics that might be included in Advanced Mathematics, far more than could be taught in the allotted time. The Commission has had to determine which of these many topics seem to be the most desirable, and make its recommendations accordingly. Following the procedure used in the discussion of Elementary and Intermediate Mathematics, both the Commission's recommendations and its reasons for making them will be given.

The Commission has decided to recommend a course to be called Elementary Functions as the core of the Advanced Mathematics program. This course can be taught either for a semester or enlarged to a full year by adding a selection of topics from the list on page 47. If Elementary Functions is given for a semester, the Commission proposes that the second semester be devoted either to introductory probability with statistical applications or to an introduction to modern algebra (fields and groups).

There are other possibilities that could be suggested. One promising alternative is an introduction to linear algebra and matrices. Indeed, any serious mathematics, creatively and imaginatively taught, would be appropriate. Schools with unusual facilities and specially trained teachers may wish to try other combinations of subject matter. The prime requisite is that such subject matter be real mathematics, challenging to the students and contributing substantially to the

development of their mathematical power.

Some schools may at first find it too difficult a task to complete the entire program of Elementary and Intermediate Mathematics by the end of grade 11. In such cases, the Commission suggests that a *minimal* four-year program consist of Elementary Mathematics, Intermediate Mathematics, and Advanced Mathematics through the Elementary Functions course.

Elementary Functions

The core of the proposed work in Advanced Mathematics consists of a modification of the traditional advanced algebra course with stress placed upon the study of certain functions: polynomials, the exponential and logarithmic functions, and the circular functions. This course may be regarded as one that picks up certain concepts from Intermediate Mathematics, builds on them, and deepens their meaning. In this course, there is also a more formal treatment of sets, relations, and functions than that given in Elementary and Intermediate Mathematics.

The study of polynomial functions includes an intuitive introduction to the notion of slope of a curve. This knowledge makes it possible to draw graphs of polynomials rapidly and with increased accuracy. Time-consuming work in locating the real roots of a polynomial equation can then be replaced by graphical methods. This approach has the dual advantage of being thoroughly practical and promoting genuine theoretical understanding. The emphasis in the work on polynomials and polynomial equations carried over into this course from the traditional advanced algebra course is to be placed on polynomials as functions, and on their functional properties. Material on roots of polynomial equations is to be regarded as an application of these properties. As already indicated, the study of polynomial functions is to be followed by a study of exponential, logarithmic, and circular functions (for an exposition of the latter, see Appendix 22, "Circular functions"). Permutations and combinations and mathematical induction are included in the course on Elementary Functions and also in Selected Topics (p. 47).

Such topics as the solution by formula of cubic and biquadratic equations, Descartes' rule of signs, Horner's method, determinants, and partial fractions are omitted.

Introductory Probability with Statistical Applications

One of the semester courses recommended for grade 12 offers an introduction to probability with statistical applications. The title is a matter of some significance. It is not intended that this course be a more elaborate presentation of descriptive statistics than the optional unit in Elementary Mathematics I (see p. 37, section X). Neither is it intended to be a course in which techniques for gathering data and similar non-mathematical aspects of statistics are discussed. This course is a course in mathematics, built around concepts of great importance for our modern technological society—the theory of probability. The objectives of the course are: (1) to introduce the student to probability concepts and to the mathematics involved in these ideas; and (2) to illustrate ways in which these concepts apply to certain common statistical problems.

The introduction of this unit is one of the more novel suggestions of the Commission.

Since the material is new and unfamiliar to many teachers, an experimental textbook giving a detailed presentation of the material recommended for inclusion in the course has been prepared.[5] This textbook has been undergoing classroom trial, but enough evidence is already at hand to indicate that the material is entirely suitable for students in grade 12. Many of them find it exciting. In the light of experimental reports a revision of the book is being issued.

So great is the current scientific and industrial importance of probability and statistical inference that the Commission does not believe valid objections based on theoretical considerations can be offered to its inclusion in the curriculum. The value of having instruction in probability with statistical applications included in college preparatory mathematics should be recognized immediately. Not only is there a great demand in a wide variety of occupations for persons with sound competence in these areas, but this material, more than most secondary school mathematics, is closely related to problems of daily living. Most people, whether they realize it or not, must frequently make decisions based upon data that are essentially statistical in character. The Commission, therefore, believes there will be general recognition of the validity of its recommendation that some of this material be taught in the secondary school; dissent can only arise, it feels, on the ground of the difficulty of carrying out the task. It is for this reason that steps have been taken to insure that the probability unit is teachable in secondary schools. Though no more intricate than some traditional topics in algebra, the theory of probability is tremendously more vital and important, and fully as valuable in developing and keeping alive algebraic skills. Since an appreciation of the ideas involved requires some maturity, we recommend that the study of probability be undertaken in grade 12. The Commission believes that many of the nation's mathematics teachers will want to familiarize themselves and their students with this important subject matter of contemporary mathematics.

Finally, a course on probability with statistical applications has an important contribution to make to the mathematical growth and development of students. The theory of probability is an example of a small axiomatic system that has remarkably extensive consequences: it takes its place beside the deductive systems recommended for geometry and for algebra. Moreover, the theory of probability provides a natural use for the important notions of sets—so central to contemporary mathematics; it also exploits and brings perspective to the traditional algebraic treatment of permutations and combinations. The binomial expansion takes on new significance when it is used to construct mathematical models for a variety of physical situations.

In addition, the proposed material provides the student with much valuable practice: in the translation of situations in real life into mathematical problems; in the construction and manipulation of functions, operations with sets, algebraic equations, and inequalities; in the use of mathematical tables (including interpolation and the extension of tables); and in the development and use of approximations, as well as of bounds. Occasions for practice with limits, maxima,

[5]Commission on Mathematics, *Introductory Probability and Statistical Inference* (revised preliminary ed., New York: College Entrance Examination Board, 1959). (This book is now out of print.)

minima, subscripts and exponents, and with the factoring and expanding of algebraic expressions, arise naturally in the work.

Apart from these specific mathematical values, the student will meet in this course the same challenge to his intellectual capabilities and his mathematical maturity that is presented by every other part of the mathematical curriculum.

Introduction to Modern Algebra

Throughout the first three years of the high school course recommended by the Commission, the field and group properties of the real numbers (i.e., commutativity, associativity, etc.) are highlighted. These properties are now studied in detail for their interrelations and consequences. Specifically, the axioms for an abstract field are extracted from the earlier experiences of the student and other properties are deduced from them. The abstract group—a still simpler system—is treated similarly. The axioms for real numbers are developed in full, and the study of transformations and their composition solidifies the concept of function. Thus algebra will be seen, indeed, as the study of mathematical structure, and this final semester's work will serve as a capstone for much of the entire secondary school course.

An additional virtue of the proposed course in groups and fields is that it makes inescapably clear to the student that deduction is not only something that is done in geometry, but is a powerful means for organizing the subject matter of other branches of mathematics. Moreover, the availability of numerous models of groups provides a fertile field for the examination of dissimilar and interesting systems.

Indeed, groups – and, to a lesser extent, fields—are so pervasive in mathematics generally that they have interesting and important applications in such diverse fields as quantum mechanics, crystallography, aesthetics, and geometry.

The subject matter of this course is of comparatively recent development in mathematics. In the early 1940's the study of groups and fields filtered down into the upper undergraduate years from the graduate school, and more recently, into the lower courses. Still more recently, carefully selected material of this sort has been found to be within the grasp of able high school students. Experience has indicated that these students have found this subject matter both challenging and interesting. A course in modern algebra in the high school can serve as an admirable means by which to bridge the gap between high school and college mathematics.

Summary

The major proposals of the Commission are outlined in the following nine points:

1. Strong preparation, *both* in concepts *and* in skills, for college mathematics at the level of calculus and analytic geometry

2. Understanding of the nature and role of deductive reasoning—in algebra, as well as in geometry

3. Appreciation of mathematical structure ("patterns") – for example, properties of natural, rational, real, and complex numbers

4. Judicious use of unifying ideas – sets, variables, functions, and relations

5. Treatment of inequalities along with equations

6. Incorporation with plane geometry of some coordinate geometry, and essentials of solid geometry and space perception

7. Introduction in grade 11 of fundamental trigonometry—centered on coordinates, vectors, and complex numbers

8. Emphasis in grade 12 on elementary functions (polynomial, exponential, circular)

9. Recommendation of additional alternative units for grade 12: *either* introductory probability with statistical applications *or* an introduction to modern algebra

Finally, as a close scrutiny of the full report will show, the Commission's recommendations embody relatively minor changes in content, but tremendously important changes in the points of view of instruction, and major changes in teaching emphases. None of these changes is simply for the sake of change. The Commission, as stated before, has tried to produce a curriculum suitable for students and oriented to the needs of mathematics, natural science, social science, business, technology, and industry in the second half of the twentieth century. This has been the overriding objective. Whatever of the old has disappeared, whatever of the traditional yet remains, whatever of the new appears, is in or out of the curriculum solely to effect necessary modification and improvement.

4. Organization: a proposed sequence for the Commission's program

This chapter presents an outline of the subject matter recommended by the Commission for a college preparatory mathematics program for grades 9 through 12. In the opinion of the Commission, this outline offers a possible, desirable, and teachable sequence. However, the Commission believes that there can be no ideal sequence, applicable to every school situation. Therefore, but only for this reason, the Commission refrains from formally recommending the sequence, and suggests rather that it be regarded as a point of departure from which modification may be made by teachers, curriculum supervisors, and textbook writers.

It must be recognized that this outline is only an outline: readers must fill in the gaps by consulting written developments of the various topics. Many topics are treated quite acceptably in numerous textbooks; others are discussed in the volume of appendices to this report (in such cases, parenthetic reference is made to the appropriate material). In particular, the discussion in the chapter immediately preceding provides much by way of amplification and elucidation.The discussion of the preceding chapter and the outline of this one are intended to supplement each other.

Throughout the teaching of the entire sequence, the Commission urges that the mathematics taught be applied to problems, real or puzzle type.

It should be noted that in the traditional curriculum for grades 10 and 11—plane geometry and intermediate algebra—the work can be, and often is, taken in the reverse order: intermediate algebra followed by plane geometry. However, in the sequence suggested by the Commission, the proposed courses for grades 10 and 11 are not thus interchangeable. It is possible to rearrange the sequence so as to defer the introduction of geometry until algebra has been more fully treated, but this will require a rather complete reorganization of the suggested outline; it cannot be satisfactorily effected merely by shifting the units from one grade to another.

In the proposed outline, the material for each year is organized into sections or topics identified by Roman numerals. It must be understood that these sections do not necessarily require equal amounts of class time.

For example, in the mathematics for grade 10, the material in section III is obviously more extensive than that of section I; the former may require more than one-third of the school year, while the latter can be covered satisfactorily in a fraction of that time.

Since the time required for the teaching of a given topic varies from class to class and from teacher to teacher, the Commission has avoided assigning time allotments to topics in the outline. This is a matter for the teacher to decide in the light of the time available, the relative importance of the topic in the course, and the nature of the class. An important principle is to do well what is done.

Attention is called to the statement on p. 31. Note that three and one-half to four years may be allotted to the teaching of Elementary Mathematics, Intermediate Mathematics, and Elementary Functions, which constitute the Commission's *minimal* four-year program. Thus, some units in grades 9, 10, or 11 may be carried over to the next grade and the slack taken up in grade 12.

Mathematics for grade 9[1] (Elementary Mathematics I)

I. Operations with simple algebraic expressions (see Appendix 1, "An introduction to algebra")

1. Notion of a collection ("set") of objects; specification of a collection—by listing its members, by stating unique conditions for membership; mainly collections constructed from natural numbers and fractions

2. Use of symbols; variable as a "placeholder"; "replacement set" of a variable; constant

3. Expressions; their description and evaluation; formulas; numerical coefficient; exponent; degree; term; factor

4. Laws of operations; use of commutative, associative, and distributive laws—reviewed for natural numbers and fractions, and extended (informally) to the transformations of simple algebraic expressions

II. Positive and negative numbers

1. Number scale; absolute value

2. The arithmetic of these numbers; operations interpreted graphically on the number scale

3. The negative of a number (additive inverse) ; the reciprocal of a number (multiplicative inverse)

4. Introduction of names "integer" and "rational number"

5. Use of variables as placeholders for rational numbers

III. Linear equations and inequalities in one variable (see the Commission's classroom unit, *Concepts of equation and inequality)*

1. Informal solution of linear equations

2. Informal solution of linear inequalities; graphical interpretation on the number scale; solution set

3. Principles of solution: equivalent equations and inequalities

IV. (Optional) Variation

1. Direct variation (e.g., $y = 3x$) ; graphical representation; informal notion of slope (simply "rise/run" from origin)

2. Inverse variation

3. Verbal problems

[1]This is essentially a course in algebra. A possible theme for the course is: *the nature and use of variables.*

V. Linear equations and inequalities in two variables (see Appendix 2, "Sets, relations, and functions")

1. Ordered pairs; solution set of a single linear equation or inequality; graphical representation

2. Systems of linear equations and inequalities (intersections of sets); graphical representation

3. Verbal problems

VI. Polynomial expressions

1. Notion of a polynomial; operations (see I, 4 above)

2. Factoring: based on distributive law, and restricted to expressions with a common monomial factor, difference of two squares, trinomials. Conventions as to what factors are admissible

VII. Rational (fractional) expressions

1. Transformations to equivalent expressions; reciprocals (multiplicative inverses)

2. Operations with rational expressions

3. Solution of equations containing rational expressions (with variable occurring in the denominator)

VIII. Informal deduction in algebra (see the Commission's classroom unit, *Informal deduction in algebra*)

1. Simple theorems on odd and even integers and properties of integers

2. Theorem: For no two natural numbers a and b is the equation $a^2 = 2b^2$ true (hence $\sqrt{2}$ is not rational).

IX. Quadratic equations

1. Condition on numbers a and b for $ab = 0$ (also conditions for $ab > 0$ and $ab < 0$)

2. Solution of quadratic equations with rational solutions (roots) (for example, $x^2 + 5x - 6 = 0$) by factoring, by completing the square, and by graphing ($y = x^2 + 5x - 6$)

3. Informal discussion of irrational numbers: solutions (roots) of $x^2 = 2$ (see VIII, 2 above, for proof that $\sqrt{2}$ is not rational); intuitive filling in of "holes" in the number scale by non-repeating, non-terminating decimals; introduction of the name "real number" (see Appendix 3, "A classroom approach to irrational numbers")

4. Solution of quadratic equations with irrational solutions (roots) by completing the square

5. Verbal problems

X. (Optional) Descriptive statistics

1. Statistical data; sets of observations and measurements

2. Collection and organization of data (charts, tables, graphs)

3. Use of single number to characterize a set of data; averages: median, mean

4. Simple measures of dispersion: range and quartiles

XI. (Optional) Numerical trigonometry of the right triangle

1. Ratio and proportion; similar triangles

2. Angle; right triangle; trigonometric ratios of acute angles (sine, cosine, tangent)

3. Tables of ratios; Pythagorean rule; review of square root

4. Solution of right triangles; computation with numbers that are approximations

5. Problems

Mathematics for grade 10[2] (Elementary Mathematics II)

I. Informal geometry

1. Review of properties of common geometric figures
2. Line segments and their measurement
3. Angles and their measurement: central angle assumption; inscribed angles

Note: The purpose of this unit is to make sure that the student has an informal, intuitive familiarity with simple geometric configurations. Material covered in grades 7 and 8 should be reviewed and extended by intuitive methods to topics listed under 2 and 3. The use of a protractor is appropriate. If students have not studied geometry informally in grades 7 and 8, discussions of this material should continue until they feel comfortable in dealing with geometric notions in an intuitive way.

II. Deductive reasoning (see Appendix 11, "A note on deductive reasoning")

1. Discussion of sentences and statements; "if-then" statements about numbers, algebraic expressions, geometric objects, and from ordinary discourse
2. Development of a concept of definitions (working descriptions) and appreciation of the role of undefined words
3. Assumptions (also called "postulates" or "axioms")
4. Notion of "proof"; hypothesis and conclusion; use of a rule of inference

III. Sequence of theorems culminating in the Pythagorean Theorem (see Appendix 12, "The first sequence of theorems")

1. Conventional assumptions, including the Parallel Postulate (but *not* superposition)
2. Preliminary theorems: supplements of equal angles are equal; complements of equal angles are equal; vertical angles are equal
3. Assumptions: Two triangles are congruent if the following parts of each are respectively equal: (1) two angles and the included side; (2) two sides and the included angle; (3) three sides
4. Theorem 1: The base angles of a triangle are equal if and only if the triangle is isosceles.
5. Theorem 2: An exterior angle of a triangle is greater than either remote interior angle.
6. Theorem 3: Two lines are parallel if and only if a transversal makes a pair of alternate, interior angles equal.
7. Theorem 4: The sum of the interior angles of a triangle equals two right angles.
8. Relations among lines and planes in two and in three dimensions: intersection, perpendicularity, parallelism; methods of representation (drawing); intuitive discussion largely, using classroom as a model to develop space perception; a few simple three-dimensional proofs
9. Assumption: A line is parallel to the base of a triangle if and only if it divides the other two sides into proportional segments.
10. Theorem 5: Two triangles are similar if two angles of one are respectively equal to two angles of the other.
11. Theorem 6: An altitude drawn to the hypotenuse of a right triangle forms two triangles, each similar to the original.
12. Theorem 7: A triangle is a right triangle if and only if the square on the largest side is equal to the sum of the squares on the other two sides.

[2]This is a course in plane and solid geometry. A possible theme for the course is: *geometry and deductive reasoning.*

13. Other related geometric material

Note: It is assumed that the development of the foregoing sequence will be supplemented by many of the usual corollaries, exercises, and theorems (including proofs of constructions) to be found in any textbook on plane geometry. The presentation should encourage creative work on the part of the student, giving him the thrill of discovering as many geometric facts and relations as he can.

14. Summary and review of the nature of deductive reasoning; (a miniature deductive system[3] proving certain properties about abstract objects may serve to bring the ideas of deductive reasoning into focus)

IV. Coordinate geometry (see Appendix 14, "Introduction to coordinate geometry")

1. Rectangular coordinates; 1:1 correspondence between points of plane and ordered pairs of real numbers

2. Directed line segments, rise and run; distance formula; slope of a segment; slope of a line; conditions for parallel and perpendicular lines

3. Mid-point formula; ratio formula for division of a line segment

4. Locus and equation (set of points and solution set)

5. Equation of straight line: point-slope form; every straight line has a first-degree equation and conversely; solution sets of linear inequalities (e.g., $y < 2x + 3$)

6. Equations of parallel and perpendicular lines

7. Equation of circle: $x^2 + y^2 = r^2$; every circle has a second degree equation (but converse *not* true); solution sets of $x^2 + y^2 > r^2$ and $x^2 + y^2 < r^2$; (optional) center not at origin

Note: Throughout unit IV, students should be given abundant practice in applying the ideas and formulas of coordinate geometry to a wide variety of situations—especially (1) proving geometric theorems algebraically, and (2) solving numerical exercises involving points and lines. (See Appendix 15, "Theorems having easy analytic proofs.")

V. Additional theorems and originals

1. A sequence on circles: theorems relating to segments of intersecting chords, secants, and tangents; to perpendicularity of tangent and radius; to measurement of inscribed and other angles associated with a circle

2. Locus in two and three dimensions (proofs where desired in two dimensions; informal treatment only for three dimensions); examples: right bisector line and plane of a line segment; points at a fixed distance from a fixed point; points equidistant from intersecting or parallel lines or planes; intersections of loci

VI. Solid geometry (see Appendix 16, "Outline of a unit in solid and spherical geometry")

1. Drawing of three-dimensional figures

2. Lines and planes; definitions and theorems

3. Angles between (1) skew lines, (2) planes, (3) line and plane; parallelism and perpendicularity

4. Solid (and plane) figures; definitions and common formulas

5. Geometry of the sphere; definitions and theorems

[3]See, for example, N. Richardson, *Fundamentals of Mathematics* (New York: The Macmillan Co., 1958) Chap. XVII.

Mathematics for grade 11[4] (Intermediate Mathematics)

I. Basic concepts and skills

1. Development of number concepts: closure of natural numbers under addition and multiplication (fractions needed for closure under division, negative numbers for closure under subtraction); closure of rational numbers under all four operations; rationals fail to solve $x^2 - 2 = 0$ (irrational numbers needed); real numbers fail to provide solution for $x^2 + 1 = 0$ (final extension to come)

2. Basic laws of operation in these systems: commutative, associative, distributive

3. Strengthening of algebraic skills (see "Mathematics for grade 9" above, sections III, p. 36; V-VII, p. 37)

II. Linear functions (see Appendix 4, "The linear function and the quadratic function")

1. Ordered pairs; sets of ordered pairs; function; functions defined by formula, table, graph

2. The linear function defined by the formula $y = mx + b$, $m \neq 0$; roles of m and b; use of m and b in sketching the graph

3. Basic property of a linear function: (change in y)/(change in x) = constant; use of this property to find a formula for a linear function given by a table of selected values

4. Direct variation ($b = 0$); graphical interpretation; problems

III. Radicals

1. Basic operations with radicals of the forms $a\sqrt{b}$, $c + a\sqrt{b}$, where a, b, c, are rational and $b > 0$ (note that "rationalizing the denominator" is included)

2. Equations involving radicals (including examples such as $\sqrt{9x-2} - \sqrt{3x-5} = 3$)

IV. Quadratic functions (see Appendix 4, "The linear function and the quadratic function")

1. The quadratic function defined by the formula $y = ax^2 + bx + c$, $a \neq 0$; roles of a, b, and c; graphical interpretation

2. Use of method of completing the square: axis of symmetry, maximum or minimum, zeros; sketching graphs; related problems on maximum and minimum

3. Discriminant; its relation to the kind of zeros, and the behavior of the graph with respect to the x-axis

V. Quadratic equations (real and complex roots)

1. Formula for roots by completing the square (roots are zeros of corresponding function); formulas for sum and product of roots

2. Discriminant; its use in characterizing roots; real numbers fail to provide a root when $b^2 - 4ac < 0$; need of extension of number system

3. Complex numbers (needed to provide roots for $x^2 + 1 = 0$ and similar equations); role of i; operations on $a + bi$ (see Appendix 5, "Introduction to complex numbers")

4. Quadratic equations with complex roots

5. Related problems

VI. Systems of equations

1. Linear-quadratic systems; graphical interpretation

2. Three linear equations in three variables; solution by successive eliminations; (optional) graphical interpretation

3. Verbal problems

[4] This is a course in algebra and elementary trigonometry. A possible theme for the course is: *real and complex numbers.*

VII. Exponents and logarithms

1. Extension of notion of exponent (zero, negative, fractional); graph of

$$y = 2^x \text{ (}x\text{ rational)}$$

2. Basic laws of exponents; their use in transforming expressions

3. Definition of logarithm; graph of

$$y = \log x \text{ (}x\text{ rational)}$$

4. Computation (moderate amount) with logarithms (stress powers and roots rather than products and quotients); basic laws; use of tables

5. (Optional) The slide rule; semi-logarithmic graphs; approximations in numerical computations

VIII. Series

1. Definitions of arithmetic and geometric series; nth term and sum of n terms (n finite)

2. Limit of a sequence (intuitive approach; see Appendix 6, "Limits")

3. Infinite geometric series; limit of s_n when $|r| < 1$

4. (Optional) Binomial series for positive, integral exponents (Binomial Theorem made plausible, but not proved)

IX. Number fields

1. Structural properties of rational, real, and complex numbers; notion of field

2. (Optional) Examples of other fields: such as

$$\{x \mid x = a + b\sqrt{3}, a \text{ and } b \text{ rational}\},$$
or the set of integers modulo 5

3. (Optional) Counter-examples: such as

the set of integers modulo 4

X. Plane vectors (see Appendix 19, "Introduction to vectors")

1. Definition

2. Displacement; multiplication by a scalar; addition and subtraction

3. Parallelogram of vectors; vector triangle

4. (Optional) Proof of geometric theorems; physical applications

XI. Coordinate trigonometry and vectors (see Appendix 20, "Coordinate trigonometry and vectors")

1. Sine, cosine, tangent of a general (directed) angle θ defined in terms of x, y, r; numerical values for special angles (0°, ±90°, ±180°, etc.; 30°, 45°, 60° and related angles in other quadrants); reduction diagrams, tables of natural functions, graphs

2. Rectangular (x, y) and polar (r, θ) coordinates of a point P (e.g., $x = -1, y = \sqrt{3}$ and $r = 2$, $\theta = 120°$); components (x, y), magnitude (r) and direction-angle (θ) of vector $\overrightarrow{OP}$ (from origin to P); conversion from (r, θ) to (x, y) and vice versa (for both P and $\overrightarrow{OP}$) by the relations $x = r \cos \theta$, $y = r \sin \theta$, and $r^2 = x^2 + y^2$, $\tan \theta = y/x$; numerical exercises

3. Complex numbers as vectors: rectangular form $x + yi$; vector addition; polar form $r(\cos \theta + i \sin \theta)$

4. Resolution and composition of vectors; physical examples

XII. Trigonometric formulas (see Appendix 21, "Trigonometric formulas")

1. Law of cosines; law of sines;

$$\text{area} = \frac{1}{2} ab \sin C$$

2. Addition formulas for cosines and sines; double-angle formulas

3. Multiplication of complex numbers in polar form; (optional) de Moivre's formula; nth roots

4. Simple identities based on

$$\sin^2 \theta + \cos^2 \theta = 1$$

and the addition and double-angle formulas; (optional) definition and slight use of secant, cosecant, cotangent

5. (Optional)

$$\pm \text{ area } OP_1P_2 = \frac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1)$$
$$= \frac{1}{2}(x_1y_2 - x_2y_1)$$

Mathematics for grade 12 (Advanced Mathematics)

Three possible programs are suggested as follows:

1. Elementary Functions, first semester; Introductory Probability with Statistical Applications, second semester
2. Elementary Functions, first semester; Introduction to Modern Algebra, second semester
3. Elementary Functions and Selected Topics: Elementary Functions enlarged to a full year by additional topics (see p. 47)

An outline of the content recommended for each of the semester courses for grade 12 is given below.

Elementary Functions

I. Sets and combinations (see Appendix 9, "The mathematics of collections of objects")

1. Review and extension of concept of set; symbolism; subsets; null set; union, intersection, complement; Venn diagrams; solution sets; graphs

2. Permutations and combinations:

$${}_nP_r,\ n!,\ {}_nC_r;$$

subsets of a given set; binomial coefficients; Binomial Theorem; (optional) extension of Binomial Theorem to negative and fractional exponents, with applications to approximations (see Appendix 7, "Permutations, selections, and the Binomial Theorem")

3. Mathematical induction (see Appendix 8, "Mathematical induction")

II. Functions and relations (see Appendix 2, "Sets, relations, and functions," and Appendix 9, "The mathematics of collections of objects")

1. Sets of ordered pairs; Cartesian set $(U \times U)$
2. Functions: definition (set of ordered pairs), domain, graphical test; methods of determining functions (table, graph, formula, rule); range of function values; notation $f(x)$ (f denotes the function); (optional) functions as mappings
3. Relations: definition, domain, range; function as a special kind of relation; graphs (equations, inequalities, including absolute value – e.g., $|x| + |y| = 1$)
4. Inverse relations and functions; graphical interpretation and tests

III. Polynomial functions

1. Brief review of linear and quadratic functions (see "Mathematics for grade 11" above, sections II, and IV, p. 40)
2. The general polynomial: definition, degree; remainder theorem, factor theorem, (optional) synthetic division; graphs of ax^n (n a small integer)

3. Slope of graph at point (x_0, y_0):

$$m(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

intuitive, numerical, and graphical discussion of this limit; tangent line

$$y - y_0 = m(x - x_0);$$

(optional) interpretation as instantaneous rate

4. Slope function m; application to curve tracing (turning points); other easy applications [maxima and minima, (optional) rates]

5. Polynomial equations: definition; graphical location of roots; graphical approximation of irrational roots; fundamental theorem of algebra (sufficiency of complex number system); number of roots; factorization; conjugate complex roots

IV. Exponential functions

1. Review of definition, properties, and graph of a^x over the rational domain (see "Mathematics for grade 11" above, section VII, p. 41)

2. Extension of definition to real domain (intuitive graphical approach); properties of function carry over

3. Exponential growth: comparison of exponential graph and polynomial graph, e.g., 2^x and x^2; graphs illustrating applications, e.g., compound interest, bacterial growth, radioactivity half-life; (optional) slope proportional to ordinate: $m = m_0 y$

V. Logarithmic functions

1. Review of definition, properties, and graph of 10^x and its inverse $\log x$ (see "Mathematics for grade 11" above, section VII, p. 41)

2. Extension to base $a (a > 0,\ a \neq 1)$; properties of function carry over

3. Graphs: $y = \log_a x$ $(a = 2, 3, 10)$; compare logarithmic and exponential graphs (reflection in $y = x$)

VI. Circular functions (see Appendix 22, "Circular functions")

1. Radian measure; $a = r\theta$; $A = \frac{1}{2}r^2\theta$; related problems

2. Definitions of $\sin x$ and $\cos x$ for real numbers x (wrapping x-axis around a unit circle); domain and range;

$$\tan x = \sin x / \cos x;$$

relation to trigonometry of angles

3. Graphs: $\sin x$, $\cos x$, $\tan x$, periodicity; $a \sin (bt + c)$, amplitude, period, phase; (optional) graphs such as that of

$$\sin 2t + 3 \cos t$$

by addition of ordinates

4. Inverse sine and inverse tangent: graphs, domain (restricted to obtain function), range

5. (Optional) Solution of trigonometric equations; evaluation of expressions such as $\sin(\arctan \frac{3}{4})$

6. (Optional) Power series for e^x, $\sin x$, $\cos x$ discussed informally ("non-terminating polynomials"); Euler's formula

$$e^{ix} = \cos x + i \sin x$$

Second-semester alternatives

The course on Elementary Functions outlined above completes the *minimal* four-year program proposed by the Commission. We hope that many schools will be able to cover this program by the end of the first semester in grade 12 for some, if not all, classes. For

these classes, one of the following three alternatives is suggested for the second semester.

Alternative 1: Introductory Probability with Statistical Applications[5]

I. The nature of probability and statistics

II. Organization and presentation of data—the frequency distribution

1. The dot frequency diagram and cumulative graph for ungrouped measurements
2. The frequency histogram and cumulative polygon for grouped measurements
3. Preview of statistical inference based on the frequency distribution

III. Summarizing a set of measurements—the mean and standard deviation

1. The mean of a set of measurements and properties of the mean
2. The standard deviation and other measures of variability of measurements
3. Computation of the mean and standard deviation for ungrouped and for grouped measurements
4. Chebyshev's theorem for a frequency distribution

IV. Intuitive approach to probability

1. Single repeated experiments with coins, dice, and cards; the notion of sample space and event; mutually exclusive and complementary events
2. The notion of probability of an event; independent events; conditional probability
3. Random numbers and their use in simple experiments

[5]This outline was fully developed in the Commission's experimental textbook, ***Introductory Probability and Statistical Inference*** (revised preliminary ed., New York: College Entrance Examination Board, 1959). (This book is now out of print.)

V. Formal approach to probability

1. More formal treatment of sample space; events, compound events; probability; probability of compound events from probabilities of simpler events; conditional probability; independence
2. Mathematical expectation

VI. The law of chance for repeated trials—the binomial distribution

1. Case of 2 and 3 trials; extension to n trials
2. Binomial probability tables and their use

VII. Applications of binomial distribution

1. Simple cases of acceptance sampling
2. Simple examples of industrial acceptance sampling plans
3. Testing from the results of many trials, a hypothetical value of the probability of success in a single trial

VIII. Using samples for estimation—sampling from a finite population

1. Using samples to make estimates of population means
2. An illustration of sampling without replacement
3. Preview of the relations between populations and samples drawn without replacement
4. The mean of the frequency distribution of sample means
5. The variance of the frequency distribution of sample means
6. Sampling with replacement
7. Mean and variance of sums of measurements drawn with replacement
8. Random sampling
9. Mean and variance of the binomial probability distribution
10. The law of large numbers

Note: The current edition of *Introductory Probability and Statistical Inference* does not include the remainder of this sequence, but stops at this point.

IX. (Supplementary) The laws of uncertainty—probability distributions

1. Chance variables
2. Probability distribution of a chance variable
3. Computing probabilities from a probability distribution
4. The close relationship between the interpretation of probability distributions and frequency distributions
5. The mean and variance of a probability distribution
6. Computing formula for variance of a probability distribution
7. Chebyshev's theorem for a probability distribution

X. (Supplementary) Relations between two variables—fitting a straight line and rank correlation

1. Introduction
2. Preview of formulas for fitting a straight line to a set of points
3. Derivation of least squares line
4. Rank correlation–definition and computation
5. Use of rank correlation to test independence of two chance variables

Alternative 2: Introduction to Modern Algebra[6] *(fields and groups)*

I. Fields

1. The system of rational numbers
   a. Recall rational numbers and their algebraic properties (from survey of "fundamental" aspects of behavior under addition and multiplication)
   b. List of properties of addition, subtraction, multiplication (i.e., axioms for a ring)
   c. Careful deduction of theorems (the "rules" of elementary algebra)
   d. Axiom on existence of inverse; definition of quotient (fraction)
   e. Deduction of properties of fractions
2. A similar system with two elements (even and odd under + and •)
3. The abstract field (keeping "+" and "•" as operations, but observing abstract nature of consequences of the axioms
4. Other fields
   a. Integers under + and • modulo 3 (show why modulo 4 gives no field; state that modulo a prime gives a field, by more advanced properties of prime numbers)
   b. Real numbers under ordinary + and •
   c. Complex numbers, defined as ordered pairs of real numbers, with axioms for a field proved

II. Ordered Fields

1. Recall properties of relation for reals
2. Axioms for ordered fields, taking "$<$" as primitive
3. Deduction of rules for manipulating inequalities (including relation $\leqq$)
4. Examples of ordered fields: rational numbers; real numbers
5. Definition of least upper bound, greatest lower bound

[6]A good first approach to some of the ideas in this course is found in *The New Mathematics* by Irving Adler (New York: John Day Co., 1958). The Commission hopes that many such books will shortly become available. The writing teams of the School Mathematics Study Group, under the direction of Dr. Edward G. Begle of Yale University, plan to produce, among other materials, some designed to acquaint teachers with ideas of modern abstract algebra.

6. Real numbers described as an ordered field in which every bounded set has a least upper bound

III. Abelian (i.e., commutative) groups

1. Introduced by examples of groups of numbers:

a. Sets of numbers closed under multiplication and division ($\{1, -1\}$, $\{1, i, -1, -i\}$, all rationals, all powers of 2, etc.)

b. Sets of numbers closed under addition and subtraction (all integers; all multiples of ⅓; all complex integers, integers modulo 3)

2. Axioms for a multiplicative abelian group

a. Restatement of axioms in additive notation

b. Proof that inverse is unique, show $b^{-1}a$ is solution of equation $bx = a$, explanation of what this means in additive case.

3. Study of examples:

a. Any set of numbers closed under subtraction is a group

b. Any set of non-zero numbers closed under product and reciprocal is a group

c. Integers modulo m under addition

d. Complex mth roots of unity

4. Isomorphism (optional)

a. Examples of isomorphic groups

(i) $\{1, -1\}$ under multiplication, with $\{0, 1\}$ (mod 2) under addition

(ii) All integers under addition with all powers of 2 under multiplication

b. Definition

c. Cyclic groups of order m (3c and d above)

IV. Transformation, composition

1. Basic ideas

a. Geometric examples of transformation groups

b. Define a transformation $T: A \rightarrow A'$ as a function

c. Composite $S \cdot T$ of two transformations: $(S \cdot T)(x) = S(T(x))$

d. Identity transformation I

2. Examples of transformation groups

a. All rotations of a regular polygon

b. All permutations of two things

c. All translations of a line, of a plane

d. All motions of a triangle or square (not commutative!)

3. Transformation groups

a. Definition of inverse transformation, $S \cdot T = I = T \cdot S$

b. Proof that inverse is unique if present

c. Define transformation groups (set of transformations of a set A, closed under composition and inverse)

V. Groups (not necessarily commutative)

1. Axioms (assuming *two*-sided identity and inverse)

2. Prove: every transformation group is a group

3. Inverses: uniqueness, inverse of product, ab^{-1} and $b^{-1}a$ as solutions of equations

4. Division algorithm $m = qn + r$ for integers (in preparation for exponents to follow)

5. Exponents and order of an element

a. Definition in abstract group of a^n, a^{-1}, a^{-n}

b. Cyclic subgroup generated by an element a

c. Order of element, using division algorithm

6. Examples: Numerous examples of transformation groups, especially those drawn from geometry

Alternative 3: Selected Topics

The two preceding alternatives are courses each requiring a full semester. This third alternative is more flexible: from it, topics may be selected for part of a semester or for a full semester. Thus, the Elementary Functions course may be allowed to extend beyond the first semester of grade 12 and then be supplemented by some of the following topics (numbered in continuation of the topics of the Elementary Functions course):

VII. Additional work on sets, functions, and relations (see Appendix 9, "The mathematics of collections of objects")

VIII. Further treatment of mathematical induction (see Appendix 8, "Mathematical induction")

IX. Further treatment of permutations, combinations, and the Binomial Theorem (see Appendix 7, "Permutations, selections, and the Binomial Theorem")

X. Probability [See *Introductory Probability and Statistical Inference*, Chapters 4, 5, 6, and (optional) 7]

XI. Additional work on inequalities and absolute values, solution sets and graphs

XII. Graphing of factorable polynomials and rational functions

XIII. Systems of equations, with emphasis on graphical interpretation

XIV. Coordinate geometry of three dimensions (planes, lines, sphere)

Note: Determinants and partial fractions are not recommended.

5. Implementation: the vital role of teacher education

In this chapter, we discuss a matter vital to the implementation of the Commission's program: the training of teachers. We comment briefly on the training of elementary school teachers, and then set forth in greater detail recommendations for the in-service and pre-service training of secondary school teachers.

Some comments on the education of elementary school teachers

All secondary school instruction in mathematics rests upon the foundations laid in elementary school. Because of this, the Commission feels that it must comment briefly about the training that elementary school teachers should receive to assure their competence in mathematics. Most of these teachers deal with arithmetic only, but some of them, especially those serving in school systems organized on the 8-4 pattern, will have to teach some version of the "junior high school mathematics" outlined earlier in this report (see p. 18, "Prerequisite mathematics"). On the sound principle that one cannot teach any subject well unless his knowledge exceeds the material he is expected to teach, even those elementary school teachers whose responsibilities include arithmetic only, should have studied mathematics beyond this point.

Considerations other than knowledge of subject matter are also, of course, extremely important in elementary teaching. Teachers should be acquainted with the findings of psychological research concerning grade placement of topics, methods of dealing with individual strengths and weaknesses, and the uses of tests. Teachers should also understand the methodology by which the child is led from exploring and discovering the meaning of number through the use of concrete materials, then to picturing what he knows and beginning to write it, and finally to working freely with abstract combinations.

However, knowledge of teaching methods and of educational psychology is no substitute for a sound understanding of such basic mathematical concepts as the number system, and its place in the mathematical scheme of things, as well as its uses in everyday affairs. The importance for the elementary teacher of training in mathematics itself is strongly evidenced in the following statement by Claude V. Courter, superintendent of schools in Cincinnati, Ohio: "More than any other subject, mathematics represents a logical, sequential system of thought

that requires careful planning of the order of subject matter and skillful presentation for effective learning. Children must learn to understand the nature of the number system early in their school life.... At all grade levels it is a guiding principle ... to present mathematics in a way that makes sense to the pupils.... Only when pupils understand the reasoning behind simple mathematical ideas are they ready to comprehend more difficult concepts."[1] Clearly, only teachers who possess an adequate background of mathematical knowledge can give children such essential understanding.

At least three years of secondary mathematics should be required: In many states it is possible for a prospective elementary school teacher to enter a teachers college after having had little or no instruction in mathematics in high school. The future teacher can then finish college without further training in mathematics—with perhaps a course in methods of teaching arithmetic, which is not a course in the study of mathematics. A substantial contribution to the improvement of mathematical instruction in America would be made by providing elementary school teachers with a background in mathematics adequate to assure their competence in teaching arithmetic.

It is true that an elementary school teacher must be competent in many fields and must have a reasonable acquaintance with still more—reading, writing, arithmetic, history, geography, art, music, science, and civics, for example. Yet among these, none possesses to the same degree as arithmetic (mathematics) that inexorably logical, sequential character that makes an adequate background in the subject matter an absolute necessity for successful and assured teaching. Thus, in the background preparation for elementary school teaching, it is imperative to devote at least as much time to curricular work in mathematics as to any other subject.

The Commission suggests that an irreducible minimum of mathematical study as essential background for teachers of arithmetic ought to be the completion in full of the first three years of a secondary school curriculum comparable to that outlined in this report. Whether required for admission by teachers colleges for students in elementary school curricula, or taught in college as part of that curriculum, no teacher whose mathematical study has been less extensive has an adequate background for teaching arithmetic in the elementary schools. Additional work in college mathematics would be highly desirable.

These recommendations of the Commission obviously cannot be implemented overnight. Appropriate adjustment of entrance requirements and curricula by teachers colleges can be accomplished only through considerable effort expended over a period of several years. Moreover, this adjustment would have to proceed in the face of the great difficulty currently experienced in attracting enough able persons into elementary teaching. The Commission nevertheless proposes adoption of the requirement of three years' study of mathematics indicated above as a vital objective to be attained as soon as possible.

For some time to come, there will of course continue to be many elementary school teachers in service who do not have what the Commission considers to be an adequate mathematical background. To repair this

[1] *Report of the Superintendent of the Cincinnati Public Schools, 1955-56.*

deficiency, we urge that appropriate programs of in-service education be established for elementary school systems. We regard the establishment of such programs as a responsibility resting inescapably upon institutions of teacher education.

It is not sufficient to ask elementary school teachers to take a course in high school mathematics taught for them by a high school teacher; moreover, elementary teachers might tend to resent programs of this character. It is necessary that this subject matter be professionalized; in particular, that its relevance to the work of the elementary schools be stressed. The appropriate person to do this is the professor of mathematics or of mathematical education in an institution devoted to teacher education. It is one thing for high school mathematics to be taught in high school to young people preparing for the profession of education; it is a very different thing to teach this same subject matter to active teachers who missed it in their preparation for the profession.

The education of secondary school mathematics teachers

More than on any other single factor, the successful carrying out of the Commission's program depends on the teachers of high school mathematics throughout the country. No curricular recommendations, however worthwhile, can be translated into classroom action except by the efforts of teachers. If they find the Commission's recommendations worthy of adoption and use them in their classrooms, then the program of the Commission can be changed from a paper formulation to a practical reality. To this end, schools must have teachers trained to teach the subject matter in the spirit of twentieth-century mathematics. These facts make it necessary for the Commission to discuss both the in-service and the pre-service education of secondary teachers of mathematics.

In-service education

Many teachers already are convinced that a reorganization of the curriculum is long overdue. This feeling was much in evidence throughout the country at meetings that were called to consider a draft of this report. These teachers wish to bring their programs and their teaching up to high standards. At the same time, they wonder how they are to acquire the training demanded by the new program.

Only a small percentage of teachers have had the up-to-date training required for the task. Only those who have begun their teaching careers recently have had an opportunity to take modern college courses, and even many of these teachers have had collegiate training of a more traditional character. Others are bound to be asking: What is contemporary mathematics? Where can we learn it? How can we use it?

The Commission offers the following suggestions with the hope that they will help teachers to prepare themselves for the task ahead and equip themselves to meet the needs of their students.

Formal programs of organized study: Teachers eager to keep abreast of developments in their field often seek to do so through intensive courses during the summer months. Unfortunately, however, the secondary school mathematics teacher who turns to a university campus for such work often finds little

to chose between two extremes: courses on methods offered by the department of education, and courses in mathematics designed for graduate students preparing for research. Neither of these meet the teacher's needs.

The advanced courses in mathematics are not suitable because in most cases teachers are returning to the campus after an absence of several years, or perhaps even a decade or two. In this period some of their prerequisite knowledge has been lost and they are no longer qualified to enter upon graduate programs they might have been fully prepared to take immediately upon graduation from college. Moreover, unlike school and lower undergraduate college courses, advanced courses generally have been modified to reflect the changes taking place at the frontiers of mathematics; this is an added reason why teachers often do not have the prerequisites to take such courses. Finally, the objective of advanced courses ordinarily is to prepare the student for research, and to do that as quickly and incisively as possible. Research usually is not the teacher's goal.

The education courses are equally unsuitable. The teacher's greatest need, in order to be prepared to teach the new curriculum, is not methodology but subject matter. To be sure, it would be most helpful if the teacher, as he studied contemporary mathematics, could be led to perceive the relevance of this subject matter to the secondary school mathematics he teaches; but in his basic preparation, there is no substitute for a solid knowledge of the elements of the new mathematics.

New courses must be designed for the needs of secondary school teachers. Departments of education and departments of mathematics must cooperate in setting up and administering such programs. Cooperation often will bring about more effective recruitment of participants, improved communication between the departments concerned, and increased understanding of their mutual responsibility for the improvement of secondary school instruction.

There have been some efforts in this direction in recent years. Some universities have recognized the necessity of providing subject-matter courses for teachers returning after a considerable absence from formal mathematical studies. Certain industries and foundations have supported conferences, summer institutes, and academic-year institutes for secondary school mathematics teachers. Funds have been provided to relieve the teacher participants of financial concerns and to enable the institute directors to engage the best scholars and teachers of mathematics from all parts of the country. Some school boards have provided grants to enable their teachers to participate in such programs. All of these activities should be encouraged; universities, industries, foundations, and school boards should be exhorted to continue and to expand their financial support of such programs and of the teachers who attend them. Yet the fact remains that the future will require a much greater and better coordinated effort than that of the past, if teachers in sufficient number are to have adequate opportunities for the training they need.

The longer instructional programs described above range in duration from six to 10 weeks and therefore allow time for two kinds of activities worth consideration. One is cooperative work on new classroom materials that the teachers can take back and try out in their own schools. The other is im-

mediate application of such classroom materials in demonstration classes conducted concurrently with the program, utilizing secondary school students brought to the campus for the purpose.

The question of academic credit for participation of teachers in such programs is troublesome. Some participants may be uninterested in credit; others must work for it in the summer and cannot attend courses unless credit is offered. Perhaps optional credit is the only solution that will satisfy both groups. Other possibilities are certificates of attendance, letters of commendation to the participant's principal or superintendent, and the like.

Conferences and similar activities: Many secondary school teachers are unable to devote a summer or longer to learning about recent developments in mathematics and how they might be incorporated into their teaching. For them, shorter organized programs like the following can and must serve as effective sources of information and stimulation.

1. Short summer conferences. Regional groups of teachers in different parts of the country have established and conducted—some for many years—summer conferences or institutes or workshops lasting from three or four days to a week or 10 days. Series of daily lectures on topics in modern mathematics generally are interspersed with single lectures on applications to science or industry (for example, computers).

2. Regular study group meetings. Some regional groups, and smaller neighborhood groups as well, have organized somewhat less formal programs of lectures, seminars, or other kinds of cooperative study that call for meetings once a week, or every two weeks, or once a month. Such a study group can be arranged by any school mathematics department of reasonable size.

3. Special university lecture series. Some university faculties are offering programs at times convenient for secondary school teachers; for example, a series of meetings on six alternate Saturdays. Lectures on topics that are manageable in the time allowed are followed by discussions of curricula. This type of conference is best suited, of course, to localities in or near large centers of population.

4. Professional society meeting sessions. Abundant opportunities are provided at national and sectional meetings of the National Council of Teachers of Mathematics, the Mathematical Association of America, and similar organizations. While most of the addresses at such meetings necessarily are brief and sporadic, they often are informative and can serve to stimulate a serious interest that is worth pursuing. Expertly conducted classroom discussions also can play an important part at meetings of this kind.

College and university mathematics faculties have a responsibility for cooperating fully with school teachers in their endeavors to bring themselves up to date in mathematics. In doing so, professors would have exceptionally fine opportunities for the kind of cooperation between departments of education and of mathematics advocated above. Moreover, shorter programs of this kind can be used effectively to acquaint not only teachers but also influential school administrators on local, county, and state levels with the needs and trends in present-day mathematics.

Finally, it should be noted that the earlier remarks concerning academic credit for

longer programs are also relevant to these shorter programs. Possibly some form of certificate of attendance is most appropriate in these cases.

Programs of continued study: Shorter organized programs and conferences such as those described in the preceding section serve admirably to indicate trends and stimulate interests in modern developments in mathematics, but they are no substitute for a suitably planned college course. Even the longer programs do not suffice. What is required is continuing effort, and the teacher who is concerned about his work and his students will find the time and energy to arrange for such study.

In some school systems, the number of teachers desiring to take the same course in mathematics is large enough to make it both convenient and economical to bring a college instructor regularly to the secondary school system. If this were done, it would be possible to give suitable modern college courses to the teachers in their own schools during the regular academic year. The school system would need to cooperate in arranging teachers' schedules so that regular course meetings would be possible under favorable conditions. Colleges should make every effort to develop suitable courses and to cooperate in such ventures. The Commission is aware that some colleges have rules against credit for courses given off-campus. We believe that such rules were made in other times to check certain evils then present, but that these rules never were intended to block a person's honest effort to improve his education, and should not now be allowed to stand as roadblocks to the meeting of a real and present necessity. There is no reason why the caliber of these college courses, specially designed for teachers, should not justify advanced credit.

Those teachers who cannot arrange a program such as the foregoing can still formulate and carry out a plan for independent study. Experience shows that such study often is most effective if pursued by a small group of teachers from the same school or from a few nearby schools meeting together for discussion, and affording each other mutual support and assistance. In particular, teachers who have attended institutes for a summer or longer can share their new knowledge, and assume leadership in further studies.

Materials available for study: In the hope of being helpful, the Commission has prepared a number of appendices, in a volume to accompany this report. In these appendices, teachers will find answers to some of their questions: How can the notion of a set be used in high school algebra? What is logically wrong with superposition as a method of proof? How do coordinates and vectors relate to the study of trigonometry? The Commission and the School Mathematics Study Group are presently planning the preparation of study guides for the use of individuals and groups. These guides would list specific readings with commentary and go beyond a bibliography, which merely indicates what to read, by indicating how to read and what to look for.

Unfortunately, there is a dearth of books on the subject matter of mathematics that are designed to meet the needs of teachers. Contemporary mathematics is for the most part embodied in advanced treatises and research journals that teachers cannot read

for the same reason they cannot take most present-day graduate courses—they do not have the prerequisites. Nevertheless, some books of special interest to teachers have been written, and more are on the way. Clear expositions of the basic subject matter of modern mathematics can be found in some of the newer textbooks for college freshmen and sophomores. These texts have the added advantage of providing secondary school teachers with information about the kinds of new material their students will encounter in college and how it will be presented there.

Periodicals such as *The American Mathematical Monthly*, and *The Mathematics Teacher*, and *School Science and Mathematics* have served and will no doubt serve increasingly as sources of articles on the subject matter and teaching of modern mathematics. Indeed, reporting as they do on experimental programs and curricular proposals, they are the chief sources of answers to questions concerning the introduction of modern mathematics in the classroom. Teachers have a right to expect that their school libraries or the public libraries in their communities, or both, will place on their shelves the books and periodicals mentioned above, as well as those recommended in the forthcoming study guide for teachers.

Pre-service education

Future graduates of institutions engaged in teacher education must not find it necessary to fill gaps in their training by methods such as those suggested for teachers now in service. It is imperative that the undergraduate programs of such institutions be modified at once to provide a sound background of study of contemporary mathematical material, and to produce teachers adequately equipped to deal with the new curricular patterns. The Commission therefore offers some suggestions as to the nature of the curriculum to be followed by a college student preparing to be a teacher of secondary school mathematics.

Mathematical program of the freshman and sophomore years: There is some reason to believe that decisions for or against teaching as a career are made during the late secondary school and early college years. If this is so, a heavy burden is placed on those courses in which a student seriously begins to prepare for the teaching profession. This is particularly true of the courses offered in the freshman and sophomore years because, of all the courses in college, these should be the ones most closely related to the school courses. It is reasonable to assume that the spirit and content of these courses can influence greatly not only the spirit and content of school mathematics but the method of instruction as well. If college courses are taught mechanistically without proper consideration of mathematical structure and present-day knowledge of the learning processes, then, in all likelihood, secondary school mathematics courses will be taught mechanistically without thought of method or content. Herein lies a heavy obligation for the colleges: they must give careful consideration to what mathematics is taught, and how it is taught, to the prospective high school teacher.

For the past few years, the Committee on the Undergraduate Program of the Mathematical Association of America has been working on the first two years of undergraduate mathematics. Their reports have in-

spired mathematics departments in many colleges and universities to undertake experimental programs. Accordingly, the Commission will not dwell longer on recommended content for the freshman and sophomore courses. New points of view and new ideas for the first two years of collegiate mathematics can be found in the reports of the Committee.[2]

Mathematics program of the junior and senior years, and of graduate work: It is difficult to differentiate between the upperclass undergraduate program and the first year of graduate work. There is no sharp dividing line between mathematics courses taken during the junior and senior years and those taken for graduate credit.

Traditionally, the undergraduate program, and, at times the graduate program too, have emphasized rather heavily the branch of mathematics commonly known as analysis. Such a program no longer is adequate. The teacher of secondary mathematics needs to know about the change in emphasis in the study of algebra that has taken place in the past quarter century. As soon as the secondary teacher learns that algebra is essentially the study of mathematical structures, he can do much to reorient algebra in his school. Similar comments can be made concerning the study of geometry, for too frequently in secondary schools it consists of a mutilated version of Euclidean geometry. This is too limited a point of view for secondary schools and particularly serious in a teacher-education program. To realize that Euclidean geometry is a special case of a more general geometry is to see geometry in an entirely different light. Here lies the key to a changed emphasis in school geometry.

Work in probability, with its applications to statistics, is also lacking in many teacher-education programs. Of course, the emphasis on statistical inference in beginning courses is of recent date, but this is no excuse for not including it in a modern program for teacher education.

It is also desirable to examine carefully the content of some courses now usually offered in many teacher-education programs. It is not unusual to require prospective teachers to take a course or two in "advanced" Euclidean geometry. It is true that prospective teachers need to re-examine Euclid from a modern point of view and to appreciate such things as the rigid requirements for construction problems laid down in Euclid. However, to follow a traditional college geometry sequence is a sacrifice of time sorely needed for other topics. Non-Euclidean geometries, topology, and the elements of differential geometry would be far more appropriate for teacher education than the usual "advanced" Euclidean geometry.

The traditional course in theory of equations also needs careful re-examination. Certain topics usually covered by such a course are more appropriately taken up in courses in modern algebra. Once this material has been placed in the proper courses, and some material omitted entirely, there is little left to justify a separate course. Theory of equations, traditionally a hodgepodge of topics,

[2] "Report of the Committee on the Undergraduate Mathematical Program," *The American Mathematical Monthly*, Vol. 62 (Sept. 1955), pp. 511-520; *Functions and Limits* ("Universal Mathematics," Part I [Lawrence, Kan.: University of Kansas, 1954; reprinted 1958]); *Elementary Mathematics of Sets, with Applications* ("Universal Mathematics," Part II [n.p.: Mathematical Association of America, 1958]); Emil Artin, *A Freshman Honors Course in Calculus and Analytic Geometry* (n.p.: Mathematical Association of America, 1957); Committee on the Undergraduate Program, *Multicomponent Methods* ("Modern Mathematical Methods and Models," Vol. I [n.p.: Mathematical Association of America, 1958]).

can well be omitted from the teacher-education program.

Teachers who return for graduate work may have had their training in an undergraduate program emphasizing traditional mathematics. Such teachers will want to take courses in modern algebra, geometry (other than Euclidean), statistics, and analysis in their graduate work. Others, who have completed their training in recent years, should continue in these four branches of mathematics, giving particular thought to those subjects that may not have received sufficient emphasis in their undergraduate studies.

While the work in algebra, geometry, statistics, and analysis and in seminars on methods of teaching will constitute a rather heavy load for the graduate years, additional time devoted to elementary symbolic logic, the history of mathematics, theory of numbers, and cognate subjects will supplement the mathematics studied in the undergraduate years. All of these subjects, properly taught, have much to offer.

Finally, the Commission would like to express a point of view concerning graduate courses in mathematics for teachers. Traditionally, the graduate program in mathematics has been designed to bring the student who has ability to the frontiers of mathematical knowledge as rapidly as possible. For those interested in mathematical research, this is as it should be.

However, these courses usually are not suitable for the secondary school teacher of mathematics. Generally, the teacher has not engaged in the concentrated study of mathematics for a period of years. When he returns to college for a few short weeks of summer work there is not time enough for him to get into the full swing of a concentrated study of advanced mathematics courses. Furthermore, there is reason to believe that the usual graduate course is not suitable, either in content or level of difficulty, because the courses for the secondary teacher should emphasize a broad understanding of the elementary aspects of a field of study and place less emphasis than usual on the mastery of advanced details or complicated proofs.

In graduate courses offered for teachers of secondary mathematics, there is need for colleges and universities to give careful consideration to the point of view from which instruction proceeds. The chief contribution that some courses (analysis, for example) have to offer at the graduate level is the development of a feeling for the spirit of modern mathematics. The teacher should understand the importance of making certain fine distinctions and the need for precise terminology and adequate symbolism, and he should develop an appreciation for mathematical rigor. All of these are more important than the precise content of the course, since the content in itself will be of less professional significance to the teacher.

Some mathematics departments do not have enough students to justify offering courses especially designed for secondary teachers. A solution must be found for this problem, since teachers will naturally gravitate to other departments for the necessary credit to earn advanced degrees if their special interests are not given careful consideration. This is not at all desirable if carried too far. The problem of developing appropriate graduate and undergraduate courses is likely to be a difficult one for many universities and colleges. However, satisfactory solutions can be achieved by plans which em-

phasize cooperation among those departments in a college or university that are interested in teacher education.

Pedagogical content of the program (undergraduate courses, student teaching, graduate courses): While mathematics courses constitute the foundation of a good teacher-education program in mathematics, there is need for courses in philosophy of education and for psychology courses that stress the emotional and mental development of the adolescent and the learning processes. Through such courses the student learns about his profession and above all gives some thought to the important problem of how children learn.

In addition to courses in philosophy and psychology, there is need for a course that deals specifically with the teaching of mathematics in the secondary schools. This course should trace enough of the history of the teaching of mathematics in the United States to provide background material for understanding present-day pressures for curriculum change. Following this, there should be a detailed discussion of some of the principal ideas of modern mathematics in the light of secondary school teaching.

From the student's point of view, the time spent under the direction of a supervising teacher is of great importance. It is the student's introduction to the "real thing." Even though it may be too much to expect teacher-education institutions in general to devote half a semester to student teaching, such a plan does exist in individual instances. Under adequate supervision, a half semester of student teaching can be a profitable assignment for the prospective mathematics teacher.

In the graduate school, the experienced teacher can profit from an advanced seminar in the problems of secondary education and the teaching of mathematics. Certainly there is enough valuable material available on the teaching of mathematics to serve as the basis of such a seminar.

Undergraduate major in mathematics: While a title cannot convey either the spirit or content of a course in mathematics, it may be helpful to those institutions that are developing or revising their programs for the education of mathematics teachers to have a list of suggested course titles. The Commission urges that, however the courses are arranged, the program should embody the concepts of contemporary mathematics, including, for example, the development of the theory of sets.

A sound teacher-education program can be developed around a major of 24 semester hours beyond the calculus. The Commission recommends that the major be earned by selecting from the following courses: differential equations, probability and statistics, modern algebra, geometry (other than Euclidean), advanced calculus, logic, history of mathematics, and theory of numbers.

In addition to the courses in pure mathematics, it is desirable for the secondary teacher to have a strong minor in at least one field that uses mathematical methods extensively. Although the physical sciences have traditionally filled this role for the mathematician, the present-day use of mathematics in the social sciences suggests many other possible choices among the contemporary applications of mathematics.

The value of courses in psychology and education should not be overlooked. Certainly all mathematics majors should have courses in such subjects as: psychology, foundations

of education, methods of teaching mathematics, and student teaching, the last of which should be the capstone of the teacher-education program.

Conclusion: While the Commission has suggested what it feels is an adequate structure for a teacher-education program, it does not wish to stereotype such programs. Certainly there is room for wide variation. However, the Commission does feel strongly that more modern approaches to the subject of mathematics must be brought into collegiate programs in mathematics. It is futile to expect the secondary schools to move in the direction of a more modern program in mathematics if the colleges do not make extensive changes in the spirit and content of their courses. In actual practice the changes in the colleges and the high schools will take place over a period of years. During this period, the colleges must supply a type of leadership that has as one of its ingredients a sympathetic understanding of the very difficult problems now confronting the secondary school teacher.

Summary

Unless a teacher of arithmetic has a mathematical background at least equivalent to the first three years of the secondary school content outlined in this report, he cannot be said to have an understanding of mathematics adequate for successful, meaningful, and assured teaching of arithmetic. Additional college work in mathematics as well as in methodology is highly desirable, if not essential.

Secondary teachers should include in their pre-service preparation courses in modern algebra, probability and statistical inference, elementary and advanced calculus, and geometry. The latter course should stress the foundations of geometry, and the study of geometries other than traditional Euclidean geometry, and should not deal exclusively or even primarily with so-called "advanced" Euclidean geometry.

Since the implementation of the Commission's program requires increased knowledge of mathematics—especially the point of view of contemporary mathematics — rather than improved methodological techniques, provision of opportunities for in-service mathematical education of teachers is imperative. Fortunately, for teachers eager to acquire the requisite knowledge, many helpful opportunities exist and more are in the making. Institutes, study groups, professional meetings, self-study, teachers' guides, and new books for teachers are examples. With the cooperation of school administrations, teachers, and colleges and universities, the vitally important task of improving in-service and pre-service education of elementary and secondary teachers of mathematics, though formidable, can be accomplished.

Colleges and universities must contribute to this end by curricular revisions of their own, particularly in the subject-matter courses designed for prospective teachers of mathematics. Revised graduate courses, designed to meet appropriate objectives of teacher education rather than to provide an introduction to advanced mathematics, are an urgent necessity.

6. Articulation: the school and the college

In view of the necessity for close articulation between secondary school and college curricula, changes in the former as recommended by the Commission call for corresponding changes in the latter. Two important aspects of this articulation are now discussed—college entrance requirements and college curricula.

College entrance requirements

For many years colleges, universities, and engineering schools have stated entrance requirements in terms of units.[1] In mathematics the custom has been not only to require a specified number of units, but actually to indicate the number of units desired in each of several subject designations within the field of mathematics. For example, a college might require one unit in algebra and one unit in plane geometry; or it might stipulate some other combination of specific subjects, such as two units in algebra, one unit in plane geometry, one-half unit in trigonometry, and one-half unit in solid geometry.

[1]A unit of secondary school work is generally understood to be a year's study of a subject in a class meeting five times a week for 40-minute or 50-minute recitation periods.

This system is open to serious objections. The basic trouble is that there are now no generally accepted definitions of the content of courses named by the old subject designations. There is no place where one can find an authoritative statement of what courses called "elementary algebra" or "solid geometry," for example, are to cover.

It was not always so. One of the factors leading to the establishment of the College Entrance Examination Board nearly 60 years ago was the fact that at that time each college issued its own definition of subject-matter requirements for entrance, and that these differed in both major and minor ways from college to college. The colleges also gave their own independent and separate entrance examinations. A primary function of the Board was to establish uniform entrance examinations based upon an established and published "Definition of Requirements" for each subject.

For many years the College Board published such a document. Such topics as "Algebra to Quadratics," "Quadratics and Beyond," "Advanced Algebra," "Plane Trigonometry," "Plane and Spherical Trigonometry" were defined carefully, the scope of the examinations so entitled was specified,

and the material to be studied in preparation for the examination was set forth in a fairly detailed syllabus. And until quite recently, a great many colleges devoted space in their catalogues to detailed specifications of content expected to be covered in college preparatory courses by candidates for admission.

Today, the College Board does not issue such definitions. A course in "intermediate algebra" (or for that matter one in "American history") includes whatever subject matter the school and the teacher wish to present. One cannot even be sure of the content of a course by examining the textbook, since omissions may be extensive and much supplementary material may be used. Only a kind of tradition, legend, or "common knowledge" indicates what the content of any particular mathematics course might be. Such indications very likely rest upon earlier College Board statements that have not been in force for many years.

In spite of this complete change in the situation, some colleges still use in their catalogues, as "requirements" for admission, subject unit designations that no longer have precise definitions. The intention and the hope, of course, is to insure standards of adequate preparation. But since the old unit designations have lost most of their validity, the supposed insurance of standards may be illusory. A candidate for admission offering a course called "intermediate algebra" is not necessarily familiar with the content the college intended to require.

If schools also put faith in such unit designations, as some do, it may act as an effective roadblock to any improvement of curricula. For example, a school attempting to conform to inflexible "requirements" may hesitate to revise its curriculum in geometry because some college to which it sends students requires in its catalogue "one unit of plane geometry." This rigid situation can nullify the recommendations of any agency seeking to improve the high school mathematics program.

Even if standardized course definitions could be brought back it would be unfortunate, for it would tend to put a strait jacket on secondary curricula in mathematics. Colleges cannot and should not dictate to secondary schools the detailed content of courses. Only cooperative action, based on the conferring of schools and colleges, can provide a satisfactory solution to this problem of college entrance requirements.

New designations needed

Mathematics is the only subject in which colleges still attempt to prescribe the detailed content of secondary school courses. Other subjects studied in preparation for college entrance for more than a year, such as English or the ancient or modern languages, do not have the year-by-year content of courses prescribed.

The basic questions in defining requirements are: "What is the content of college preparatory mathematics?" and "How much of it should be required?"

The Commission would answer the first question as follows. College preparatory mathematics should include topics selected from algebra, geometry (demonstrative and coordinate), and trigonometry—all broadly interpreted. The point of view should be in harmony with contemporary mathematical thought; emphasis should be placed upon basic concepts and skills, and upon the prin-

ciples of deductive reasoning regardless of the branch of mathematics from which the topic is chosen. In every case, the standard of substance and content should be commensurate with that of the courses outlined in Chapter 4. Courses designed for other purposes (e.g., consumer mathematics, business mathematics, shop mathematics) are not acceptable.

Answering the second question on the basis of such content, the Commission advocates that colleges designate in their catalogues the amount of secondary school mathematics required for admission simply in terms of the length of time spent in study—as is done in every other subject.

As nomenclature, the Commission suggests the general use of the terms it has previously defined in this report (p. 20) : for the first and second high school years, Elementary Mathematics; for the third high school year, Intermediate Mathematics; and for the fourth year, Advanced Mathematics. Since the term Elementary Mathematics covers two years, it might be divided into Elementary Mathematics I and Elementary Mathematics II if necessary, desirable, or appropriate. A college requiring three years of study would indicate its requirements as Elementary and Intermediate Mathematics; a college requiring four years would stipulate Elementary, Intermediate, and Advanced Mathematics.

An institution could of course use only the term "Mathematics" with an indication of the number of years of study required. Some interest has been expressed in this method of designation.

Not waiting for the publication of this report, the Commission early in 1957 sent letters on this problem of requirements to all College Board member colleges, and also to all regionally accredited colleges in the United States with enrollments of 500 or more students. The replies to these letters indicate a general willingness to consider the Commission's recommendations. The Commission urges that they be adopted with all possible speed.

College Board test requirements

In addition to secondary school mathematics courses, a number of colleges—particularly engineering colleges—require for entrance either the Intermediate Mathematics or the Advanced Mathematics Achievement Tests of the College Board. Readers will of course notice that the names of these tests are the same as the names suggested by the Commission for courses presenting the mathematical content it recommends for grades 11 and 12. Moreover, the Intermediate Test is intended as a test of work through grade 11 and the Advanced Test, of work through grade 12.

However, despite the similarity in names and grade levels, the mathematics represented in the tests and the mathematics recommended by the Commission are *not* the same at present. As has been said, the tests are not being suddenly changed to correspond with the Commission's recommendations. Those recommendations are instead being taken as a guide for a gradual change in the examinations. This change should extend over a period of years and will proceed as mathematics programs actually being taught in the schools are changed.

Inasmuch as some schools are already teaching some of the content recommended by the Commission, this change in the tests

has been begun in order to make the tests fair to students in these schools. As in the past, the distribution of types of content and of the difficulty level of questions within each type is and will continue to be arranged in the tests to minimize the effects of differences in curricula. Thus, on the one hand, the tests are not being subjected to changes that would make them any less widely applicable than they have been; on the other hand, they are not being kept so rigidly unchanged as to impede adoption of the Commission's recommendations.

In view of this, colleges now using the mathematics Achievement Tests need make no change in their requirement of the tests to facilitate adoption of the Commission's proposals. Neither requiring the tests nor not requiring them will affect realization of the Commission's aims.

College curricula

The colleges have contributed to the high school curriculum problem by their inertia; they must now contribute to its solution by improving their own programs. They have been lagging and uncertain in dealing with the implications of contemporary mathematics for college curricula. If the colleges had faced these implications with vision and resolution, the necessary reform at the secondary school level might have started long since.

But there is no point in dwelling on what might have been. The future must now learn from the past, and colleges as well as high schools must accept their responsibility to provide mathematics programs suited to present-day needs. Many college curricula fail to meet this requirement.

Too often the basic freshman course consists of trigonometry and college algebra. The Commission's program is based on the belief that the normal college freshman course for students presenting four years of secondary school mathematics should consist of analytic geometry and calculus, or material of similar caliber. For students offering three years, a less demanding freshman course should be available, but this too should be of real college character. It would be inexcusably wasteful for colleges to ask freshmen who have completed three or four years of the program recommended in this report to take traditional courses in college algebra and trigonometry.

The Commission sees no hope of effective curricular reform at the secondary school level unless this reform is supplemented by a related effort at the college level. Fortunately, and commendably, at many colleges such an effort is already under way.

Summary

College entrance requirements should be restated in broad and significant terms designed to encourage schools to introduce curricula and courses oriented toward the development of mathematical power, insight, and understanding, rather than toward coverage of formalized material. It may safely be assumed that College Board tests will be so constructed as to minimize the effects of differences in curricula, and neither to impede nor unduly hasten the adoption of modernized programs.

Colleges have had a heavy responsibility laid upon them by the Commission. They must revise their freshman courses so that

freshmen who enter college having completed three and one half or four years of the Commission's program are placed in a substantial calculus course, and those with three years of preparation, in a modified calculus course or some other appropriate course of college level. The traditional freshman courses will not suffice.